



ELSEVIER

Journal of Geometry and Physics 19 (1996) 207–245

JOURNAL OF
GEOMETRY AND
PHYSICS

Geometric quantization of reduced cotangent bundles

M.A. Robson^{1,2}

*Department of Applied Mathematics and Theoretical Physics, University of Cambridge,
Silver Street, Cambridge, CB3 9EW, UK*

Received 2 June 1994; revised 4 April 1995

Abstract

The method of geometric quantization is applied to the symplectic space obtained by Marsden–Weinstein reduction of a cotangent bundle T^*N . Specifically it is assumed that the symmetry [gauge] group acts freely on N so that N is a principal H -bundle over the assumed Riemannian manifold $Q = N/H$. The reduced phase space obtained is the same of that for a charged particle moving on Q in an external Yang–Mills gauge field which is given by a connection on $N \rightarrow Q$. An explicit map is found from a subalgebra of the classical observables to the corresponding quantum operators. These operators are found to be the generators of a representation of the semi-direct product group, $\text{Aut } N \times C^\infty(Q)$. A generalized Aharonov–Bohm effect is shown to be a natural consequence of the quantization procedure. In particular, the role of the connection in the quantum mechanical system is made clear. The quantization of the Hamiltonian is also considered.

Additionally, our approach allows the related quantization procedures proposed by Mackey and by Isham to be fully understood.

Keywords: Geometric quantization; Marsden–Weinstein reduction; Constrained systems; External gauge fields

1991 MSC: 81S10, 58F06

PACS: 03.65Bz

1. Introduction

Constrained mechanical systems make up an important category of classical dynamical systems. We consider the case where the constrained system is described by a symplectic manifold S (the “unconstrained” phase space of the system) together with a Lie group H ,

¹ Supported by an E.P.S.R.C. studentship.

² Present address: Sema group UK Ltd, Trafalgar House, Richfield Avenue, Reading RG1 8QA, UK.

which has a Hamiltonian action on S , and a corresponding equivariant momentum map $J_H : S \rightarrow \mathfrak{h}^*$ where \mathfrak{h} is the Lie algebra of H and \mathfrak{h}^* is its dual. The constraints are given by $J_H = \mu$ for some fixed $\mu \in \mathfrak{h}^*$. Subject to certain technical assumptions, the reduced phase space of the system (i.e., the true phase space of the system in which the constraints are automatically satisfied) is then a quotient manifold of $J_H^{-1}(\mu)$ and inherits a symplectic structure from S [35]; this particular method of identifying the reduced phase space is called Marsden–Weinstein reduction. A key point is that this quotient manifold is symplectomorphic to a symplectic leaf in the Poisson manifold S/H [32,21]. The Poisson bracket on S drops to one on S/H (since H acts symplectically on S) and this defines the Poisson structure on S/H .

We concentrate on the case where S is a cotangent bundle T^*N . We assume that H acts freely on N so that we may consider $(N, Q, H, \pi_{N \rightarrow Q})$ to be a principal fibre bundle with total space N , base space $Q = N/H$, projection $\pi_{N \rightarrow Q}$, and where the Lie group H acts on the right of N . Thus H acts on $S = T^*N$ by cotangent lift and there is always an equivariant momentum map for this action [1]. The reduced phase space is then a symplectic leaf in $(T^*N)/H$. For the special case $N = H$ it is well known that the reduced phase space is just a coadjoint orbit in \mathfrak{h}^* . Our results can be regarded as an extension of this special case to the general one. Our method centres around finding a rich group structure on the reduced phase space which is based upon the relatively natural construction of commuting automorphisms.

One important physical interpretation, originally due to Sternberg, of this type of constrained system, is well known in the context of a charged particle moving on Q in the presence of an external Yang–Mills field with gauge group H [46,48,36,14]. Specifically $S/H = (T^*N)/H$ is the “universal phase space” of the particle. There is a one-to-one correspondence between the symplectic leaves of S/H and the coadjoint orbits in \mathfrak{h}^* . Each of the latter represents a different possible charge of the particle so that S/H , which is foliated by its symplectic leaves, is composed of the phase spaces corresponding to every possible charge. With respect to this example we often refer to Q as the configuration space and H as the gauge group.

Naturally the construction of the quantum mechanical system corresponding to a constrained mechanical system has aroused much interest. Recall that quantization tries to associate to each classical system (described by a symplectic manifold S) a Hilbert space \mathcal{H} of quantum states and a map from the space of classical observables (smooth functions on S) to the space of symmetric operators \mathcal{O} on \mathcal{H} . Each classical observable $f \in C^\infty(S)$ should correspond to an operator $\hat{f} \in \mathcal{O}$ such that:

- (Qi) the map $f \rightarrow \hat{f}$ is linear (over \mathbb{R});
- (Qii) if $f = \mathbb{1}_S$, then $\hat{f} = \mathbb{1}_{\mathcal{H}}$, where $\mathbb{1}$ denotes the identity operator;
- (Qiii) if $\{f_1, f_2\} = f_3$ then $[\hat{f}_1, \hat{f}_2] = i\hbar \hat{f}_3$.

Additionally, some sort of irreducibility condition is also imposed. When $S = T^*Q$ is a cotangent bundle the operators \hat{q} and \hat{p} corresponding to $(q, p) \in T^*Q$ are required to act irreducibly whilst when S is a coadjoint orbit the map $f \rightarrow \hat{f}$ must give an irreducible representation of the generators of the symmetry group. In order to meet this requirement of irreducibility, restrictions are imposed, in all quantization schemes, on the class of

observables that can be quantized. However, for a *general* symplectic manifold there appears to be, in the literature, no definite statement of the irreducibility requirements. We shall find though, that our method associates with quantization a representation of a Lie group which is, in general, irreducible, thus meeting any reasonable irreducibility requirement.

Much work [12,40,10,11,6] has been done on the quantization of the reduced phase spaces of constrained systems and of a particle in a gauge field, most notably by Landsman for the case of homogeneous configuration spaces [29] using induced representations and for the general case [30,27] using Rieffel's notion of "strict deformation quantization" and Rieffel induction respectively. However, these approaches suffer from the inability, in general, to quantize classical observables which are unbounded. Of course, geometric quantization places a restriction on the observables which can be quantized. However, we can quantize certain unbounded observables, such as momentum and position, and this is superior to the other approaches because the physical observables which are of interest are usually unbounded.

The method of geometric quantization has so far been restricted to comparing the quantization obtained by first solving the constraints (i.e., reduction) and then quantizing the reduced space or quantizing the extended phase space and then imposing the constraints at a quantum level. However, in the setting that we work in, it was found [12] that the two approaches were equivalent only if the reduced phase space was symplectomorphic to a cotangent bundle and thus the geometric quantization of the reduced phase space was restricted to this rather special case. The question of whether reduction and quantization commute for arbitrary quantizable symplectic spaces is discussed in [42]. For the case in hand of cotangent bundles it is shown that the two procedures do commute *provided* the "unconstrained" system is taken to be $T^*N \times \mathcal{O}$, where $\mathcal{O} \subset \mathfrak{h}^*$ is the coadjoint orbit containing the point μ . (The reduced space arising from consideration of $T^*N \times \mathcal{O}$ is symplectomorphic to the original reduced space.)

Another notable contribution is in the area of homogeneous configuration spaces Q . This study was initiated by Mackey [31] and was extended by Isham [20], who used a group-theoretic approach to identify a particular semi-direct product group \mathcal{G} which acted on the phase space T^*Q of the system. Quantization then corresponded to assigning quantum operators to be generators of an irreducible unitary representation of the group \mathcal{G} . However, as in general, there is more than one such representation of this group, many different inequivalent quantum systems arise from the study of the same configuration spaces. We will see that these correspond to the geometric quantization of different symplectic leaves of $(T^*G)/H$ where G is a Lie group ($H \subset G$) so that G/H is the homogeneous space Q . (Note that $T^*Q \subset (T^*G)/H$.) Indeed, the underlying motivation for this paper was the anticipation of this result, which was based upon two previously known results. Firstly, it has been shown [34] that the symplectic leaves of $(T^*G)/H$ are symplectomorphic to certain coadjoint orbits in the dual of the Lie algebra of \mathcal{G} . Whilst secondly, Rawnsley [41] has shown that the geometric quantization of these orbits leads to the induced representations upon which Mackey theory is based.

One important feature of Isham's approach was the use of a momentum map to relate the classical observables with their quantum operator counterparts. Specifically if \mathcal{G} is a Lie

group, with Lie algebra $\mathcal{L}(\mathcal{G})$, which acts on the symplectic manifold M and $J : M \rightarrow \mathcal{L}(\mathcal{G})^*$ is a corresponding equivariant momentum map then, given a representation π of \mathcal{G} , a “quantizing map” Q_\hbar can be given which relates classical observables to quantum operators. Explicitly,

$$Q_\hbar(\hat{J}(A)) = \hbar d\pi(A), \quad (1.1)$$

where $\hat{J} : \mathcal{L}(\mathcal{G}) \rightarrow C^\infty(M)$ is defined by $\langle J(m), A \rangle = (\hat{J}(A))(m)$ and $d\pi$ is the derived Lie algebra representation, where we are following the convention that

$$d\pi(A) = i \left. \frac{d}{dt} \pi(e^{tA}) \right|_{t=0}. \quad (1.2)$$

The general properties of a momentum map then ensure that condition (Qiii) is automatically satisfied. The subclass of observables that can be quantized by Q_\hbar is clearly $\{\hat{J}(A) : A \in \mathcal{L}(\mathcal{G})\}$. Clearly, this approach hinges on the ability to choose \mathcal{G} and π correctly.

This paper uses the geometric quantization framework of Kostant and Souriau to give a complete quantization of the constrained mechanical system whose reduced phase space is [symplectomorphic to] a symplectic leaf in $(T^*N)/H$. In particular, our only assumptions are that the space N has a Riemannian structure with an H -invariant metric (so that Q inherits a metric from N) and that the gauge group H is a connected and compact Lie group. We are able to combine naturally the group-theoretic and geometric quantization approaches, finding on the way how each sheds light on the other. In particular, we are able to present our results in the language of representations so that the quantum operators are given as generators of a representation of a Lie group, together with a corresponding momentum map which explicitly links the quantum operators with their classical observable counterparts in the manner described above. Thus, no knowledge of geometric quantization is required in order to appreciate the results found.

Our presentation relies very heavily on the combination of the symplectic formulation of constrained mechanical systems with the method of geometric quantization. Since no one source adequately presents both theories in detail and manner needed, a short review of both is given in Sections 2 and 3 respectively. In particular, Section 2.4 gives a new result regarding the action of the semi-direct product group $\mathcal{G} = \text{Aut } N \ltimes C^\infty(Q)$ of fibre preserving diffeomorphisms of N and smooth functions on Q on the phase space of the reduced system. This group action has a corresponding momentum map and the idea is to quantize in the style of (1.1). Indeed Section 4 can be regarded essentially as justifying this choice of \mathcal{G} and showing which representation π of \mathcal{G} is to be chosen in the right-hand side of (1.1). Section 4.7 explicitly compares our approach with that of Isham’s [20] for homogeneous spaces.

Section 3 reviews very briefly the method of geometric quantization. Finally, Section 4 forms the heart of the paper. Using geometric quantization the reduced phase space is quantized. We find that, using a particular polarization, the subclass of observables that can be quantized is the same as that predicted by use of the group $\mathcal{G} = \text{Aut } N \ltimes C^\infty(Q)$. We then show that the corresponding quantum operators are generators of a representation π of \mathcal{G} , the choice of π depending on which symplectic leaf of $(T^*N)/H$ we are quantizing

on. Along the way we find that the Aharonov–Bohm effect is a natural consequence of our quantization and also that the nonintegrable phase factor of Wu and Yang [51] appears in the analogous result for the case when the gauge group is non-Abelian. Using the results of [30] we are able to give a Hamiltonian for the quantum system which then completes the quantization of the constrained system.

2. Constrained mechanical systems

2.1. Dual pairs and momentum maps

We start by reviewing the basic ideas of dual pairs and momentum maps which provide great insight into the theory of Marsden–Weinstein reduction. The main references for this subsection are Weinstein [49], Choquet-Bruhat and De Witt-Morette [7, Ch. 12] and Abraham and Marsden [1]. Also see the article by Adams and Ratiu in [15] for a review of dual pairs in connection with the integrability of point vortex motion.

A useful idea is the notion of a *realisation* of a Poisson manifold M . This is a symplectic manifold S together with a *Poisson map* $J : S \rightarrow M$. A Poisson map is one which preserves the Poisson bracket, i.e.,

$$\{J^*F, J^*G\}_S = J^*\{F, G\}_M \quad \forall F, G \in C^\infty(M).$$

We are interested in the case when the fibres $J^{-1}(m)$, $m \in M$, define a foliation Φ of S in such a way that S/Φ is a manifold and so if $\pi : S \rightarrow S/\Phi$ is the canonical projection, the space $\pi^*C^\infty(S/\Phi)$ is a Lie subalgebra of $C^\infty(S)$ and coincides with $J^*C^\infty(M)$. We denote the functions constant on the leaves of Φ by \mathcal{F}_Φ and the functions which Poisson commute with all elements of \mathcal{F}_Φ by \mathcal{F}_{Φ^\perp} , i.e., symbolically $\{\mathcal{F}_\Phi, \mathcal{F}_{\Phi^\perp}\} = 0$. It can be shown that this defines a foliation Φ^\perp of S and with the assumption that S/Φ^\perp is a manifold, the \mathcal{F}_{Φ^\perp} are functions constant on the leaves of Φ^\perp . We call \mathcal{F}_Φ and \mathcal{F}_{Φ^\perp} *polar* to each other.

A *dual pair* is where we have two Poisson manifolds M_1 and M_2 and a symplectic manifold S with Poisson maps J_1, J_2 between S and each M_i

$$M_2 \xleftarrow{J_2} S \xrightarrow{J_1} M_1$$

and \mathcal{F}_{Φ_1} and \mathcal{F}_{Φ_2} are polar to each other. The dual pair is called *full* if J_1 and J_2 are both submersions. However, if J_1 and J_2 have constant rank then $J_1(S)$ and $J_2(S)$ are Poisson submanifolds and $J_2(S) \xleftarrow{J_2} S \xrightarrow{J_1} J_1(S)$ is a full dual pair. Assuming this to be the case, then the key result is that we can define a bijection between the symplectic leaves of M_1 and M_2 (assuming that J_1 and J_2 have connected fibres). Specifically, if $J_1^{-1}(m)$ is connected, then $J_2(J_1^{-1}(m))$ is a symplectic leaf of M_2 . In general, $J_2(J_1^{-1}(m))$ will be a union of [connected] symplectic leaves of M_2 .

For the case where a Lie group G acts symplectically on [the left of] a symplectic manifold S (i.e., the Poisson bracket is invariant under the action of G) such that $G \backslash S$ is a manifold we can often find a *momentum map* J such that we have the dual pair

$$\mathfrak{g}_+^* \xleftarrow{J} S \xrightarrow{\pi^G} G \backslash S, \tag{2.1}$$

where π^G is the projection map $S \rightarrow G \backslash S$ and \mathfrak{g}_+^* is the dual of the Lie algebra of G with the “+” Lie–Poisson structure. This means we can find the symplectic leaves of $G \backslash S$ using the above result.

Recall that the function groups corresponding to J and π^G must be polar to each other. This gives us our first condition on J . Denote the infinitesimal generator of the left action of G on S by ξ , i.e.,

$$(\xi(X)f)(m) = \left. \frac{d}{dt} f(e^{tX} \cdot m) \right|_{t=0}, \quad X \in \mathfrak{g}, m \in S, f \in C^\infty(S), \tag{2.2}$$

where we are denoting the action of $x \in G$ on $m \in S$ by $x \cdot m$. (Note that, if G acts on the right on S , then $e^{tX} \cdot m$ in (2.2) is replaced by $m \cdot e^{tX}$ and \mathfrak{g}^* now has the “–” Lie–Poisson structure.) Now define $\hat{J} : \mathfrak{g} \rightarrow C^\infty(S)$ to be the restriction of J^* from $C^\infty(\mathfrak{g}^*)$ to \mathfrak{g} (regarding $\mathfrak{g} \subset C^\infty(\mathfrak{g}^*)$, i.e., $X(\theta) \equiv \langle \theta, X \rangle$ for $X \in \mathfrak{g}$ and $\theta \in \mathfrak{g}^*$). So explicitly

$$\hat{J}(X)(m) = \langle J(m), X \rangle. \tag{2.3}$$

Then the first condition is that \hat{J} must satisfy

$$\{f, \hat{J}(X)\} = \xi(X)f \quad \forall f \in C^\infty(S), \forall X \in \mathfrak{g}. \tag{2.4}$$

Note that we define the Hamiltonian vector field ξ_f of $f \in C^\infty(S)$ by $\xi_f g = \{g, f\}$ for $g \in C^\infty(S)$, so that (2.4) can be written $\xi_{\hat{J}(X)} = \xi(X)$. Secondly J must also be a Poisson map; this is achieved by the condition that J must be equivariant, i.e.,

$$J(x \cdot m) = \pi_{\text{co}}(x) \cdot J(m). \tag{2.5}$$

Here $\pi_{\text{co}}(x) \equiv \text{Ad}_{x^{-1}}^*$ denotes the coadjoint action. This last condition implies that

$$\{\hat{J}(X), \hat{J}(Y)\} = \hat{J}([X, Y]), \tag{2.6}$$

i.e., \hat{J} preserves the Lie algebra structure. With these two conditions it can be shown that we have a dual pair as described.

Now, assuming that G is connected, the symplectic leaves of \mathfrak{g}^* are coadjoint orbits. Also note that (2.5) implies that $\pi^G(J^{-1}(\pi_{\text{co}}(\mathfrak{g}) \cdot \mu)) = \pi^G(J^{-1}(\mu))$. Thus, assuming the fibres $J^{-1}(\mu)$ are connected, the symplectic leaves of $G \backslash S$ can be written $P_{\mathcal{O}_\mu} = G \backslash (J^{-1}(\mathcal{O}_\mu))$ where \mathcal{O}_μ is a coadjoint orbit in \mathfrak{g}^* . The symplectic form $\Omega_{\mathcal{O}}$ on $P_{\mathcal{O}}$ is given, e.g., [33], via

$$j_{\mathcal{O}}^* \Omega = \text{pr}^* \Omega_{\mathcal{O}} + J_{\mathcal{O}}^* \omega_{\mathcal{O}+}, \tag{2.7}$$

where $j_{\mathcal{O}} : J^{-1}(\mathcal{O}_\mu) \rightarrow S$ is the inclusion, Ω is the symplectic form on S , $\text{pr} : J^{-1}(\mathcal{O}_\mu) \rightarrow P_{\mathcal{O}_\mu}$ is the projection π^G acting on $J^{-1}(\mathcal{O}_\mu)$; and where $J_{\mathcal{O}} = J|_{J^{-1}(\mathcal{O}_\mu)} : J^{-1}(\mathcal{O}_\mu) \rightarrow \mathcal{O}_\mu$ and $\omega_{\mathcal{O}+}$ is the symplectic form on the coadjoint orbit.

There is an alternative expression for the symplectic leaves of S/G . If G_μ denotes the isotropy group of μ , i.e.,

$$G_\mu = \{g \in G : \pi_{\text{co}}(g) \cdot \mu = \mu\}, \tag{2.8}$$

then the symplectic leaf $P_{\mathcal{O}_\mu} = \pi^G(J^{-1}(\mu)) \simeq G_\mu \backslash J^{-1}(\mu) = P_\mu$. This process of identifying P_μ or $P_{\mathcal{O}_\mu}$ is called Marsden–Weinstein reduction. The symplectic structure Ω_μ on P_μ is given by $i_\mu^* \Omega = \pi_\mu^* \Omega_\mu$, where $i_\mu : J^{-1}(\mu) \rightarrow P_\mu$ is the inclusion map and $\pi_\mu : J^{-1}(\mu) \rightarrow P_\mu$ is the projection map.

There is an important general result regarding the actions of a group G on a manifold Q . Specifically the induced action of G on the cotangent bundle T^*Q (by cotangent lift) is symplectic with respect to the canonical symplectic form $\sigma_0 = -d\theta_0$ on T^*Q . Here θ_0 is the canonical one-form defined by

$$\langle \theta_0, v \rangle_{\beta_q} = \langle \beta_q, \pi_* v \rangle_q \quad \forall v \in T_{\beta_q}(T^*Q), \tag{2.9}$$

where $\beta_q \in T_q^*Q$ and $\pi : T^*Q \rightarrow Q$ is the canonical projection. An equivariant momentum map for the action of G on T^*Q is given by $J : T^*Q \rightarrow \mathfrak{g}^*$ with

$$\langle J(p_q), X \rangle = \langle p_q, \xi(X) \rangle, \quad X \in \mathfrak{g}. \tag{2.10}$$

There is a slightly different approach, at least in the language used, to finding the symplectic leaves of $G \backslash S$. In this terminology, e.g., [50], the submanifold $J^{-1}(\mu) \subset S$ is called a *presymplectic manifold*. It has a two-form σ' given by just restricting the symplectic form on S to $J^{-1}(\mu)$. The *characteristic distribution* of σ' is

$$K_m = \{X : i_X \sigma' = 0\} \subset T_m(J^{-1}(\mu)) \tag{2.11}$$

provided the dimension of K_m remains constant for all $m \in J^{-1}(\mu)$. It follows that K is integrable and if $M = J^{-1}(\mu)/K$ is a manifold then σ' projects onto a well defined symplectic structure σ on M . This new symplectic space (M, σ) is called the *reduction* of $(J^{-1}(\mu), \sigma')$.

The link between the symplectic space M and the symplectic leaves $J^{-1}(\mu)/G_\mu$ found earlier, is that, in general, $K = (G_\mu)_0$, the identity component of G_μ . When G is compact or semi-simple, the isotropy group G_μ is connected [14] so $K = G_\mu$ in agreement with our earlier approach. For the special case $S = T^*G$, each $\beta \in T_x^*G$ can be identified with the one-form $\lambda_x^* \beta \in T_e^*G \simeq \mathfrak{g}^*$ where λ_x^* is the pullback of the left action $\lambda_x y = xy$. This gives us the [left] parallelization

$$\begin{aligned} T^*G &\rightarrow G \times \mathfrak{g}^* \\ \beta &\rightarrow (x, \lambda_x^* \beta)_L. \end{aligned} \tag{2.12}$$

Let $\{d^a\}$, $a = 1, \dots, d_G = \dim G$, be a basis of \mathfrak{g}^* and define $\theta^a(x) = \lambda_{x^{-1}}^* d^a$ so $\{\theta^a(x)\}$ form a basis for the left invariant one-forms on G . Any element $\beta \in T_g^*G$ can be expanded as $\beta = p_a \theta^a(x)$ and in the above parallelization this corresponds to $\beta \rightarrow (x, p_a d^a)_L$. Hence we can use the $\{p_a\}$ as coordinates on T_g^*G which are globally valid. For future reference, the canonical one-form in this coordinate system is

$$\theta_0(x, p) = p_a \theta^a(x). \tag{2.13}$$

(Note that similarly there is a right parallelization of $T^*G \simeq G \times \mathfrak{g}^*$ via $\beta \rightarrow (x, \rho_x^* \beta)_R$ where ρ denotes the right action of G on G .)

If G acts on the right of $S = T^*G$ then it follows from (2.10) that an equivariant momentum map is given by $J_R : T^*G \rightarrow \mathfrak{g}^*$ with $J_R(x, p)_L = p$. Clearly, $J^{-1}(\mu) = G \times \{\mu\} \simeq G$. Thus G can be regarded as a presymplectic manifold with $K_x = \{L_V(x) : V \in \mathfrak{g}_\mu\}$ where \mathfrak{g}_μ denotes the Lie algebra of G_μ and L_A denotes the left invariant vector field on G generated by $A \in \mathfrak{g}$, i.e., for $f \in C^\infty(G)$,

$$(L_A f)(x) = \left. \frac{d}{dt} f(xe^{tA}) \right|_{t=0}. \tag{2.14}$$

The reduction of G gives, as expected, the manifold $G/G_\mu \simeq \mathcal{O}_\mu$ when $(G_\mu)_0 = G_\mu$.

2.2. Mechanical H -systems

We work in the setting of what Smale [44] calls a simple mechanical H -system. This means that we have a symplectic manifold T^*N , together with a right action of H on N (H acts on T^*N by cotangent lift), a Riemannian metric on N which is H -invariant and a Hamiltonian, $H_0 : T^*N \rightarrow \mathbb{R}$, of the form

$$H_0(n, p) = \frac{1}{2} \|p\|_n^2 + V(n), \tag{2.15}$$

where $\|\cdot\|_n$ is the norm induced on T_n^*N , and where V is an H -invariant potential. We assume that H acts freely on N so that we can regard $N \rightarrow Q = N/H$ as a principal fibre bundle as described in Section 1. Now H_0 is H -invariant so Marsden–Weinstein reduction gives a reduced Hamiltonian system on the reduced space P_μ (or alternatively on $P_{\mathcal{O}_\mu}$). Marsden [33] has given an explicit realization of P_μ as a submanifold of $T^*(N/H_\mu)$, where H_μ is the isotropy group of H defined in (2.8). The essential part of this realization is what Marsden calls the mechanical connection.

The *locked inertia tensor* $\mathbb{I}(n) : \mathfrak{h} \rightarrow \mathfrak{h}^*$ is defined at each $n \in N$ via

$$\langle \mathbb{I}(n)X, Y \rangle = \langle \xi_n(X), \xi_n(Y) \rangle, \tag{2.16}$$

where $\xi(X)$ denotes the infinitesimal generator of the action of \mathfrak{h} on T^*N . We identify \mathbb{I} with the metric on \mathfrak{h} . Let $FL : TN \rightarrow T^*N$ be the *Legendre transformation* for the simple mechanical H -system (e.g., see [1]). The *mechanical connection* $\alpha : TN \rightarrow \mathfrak{h}$ is defined by

$$\alpha(n, v) = \mathbb{I}(n)^{-1}(J(FL(n, v))), \tag{2.17}$$

where $J : T^*N \rightarrow \mathfrak{h}^*$ is the momentum map for the action of H on T^*N . As mentioned in the previous section, the momentum map $J : T^*N \rightarrow \mathfrak{h}^*$ for the [right] action of H on T^*N is provided by means of (2.10). In our present notation,

$$\langle J(p_n), X \rangle = \langle p_n, \xi(X) \rangle, \quad X \in \mathfrak{h}^*. \tag{2.18}$$

The term mechanical connection is used because α defines a connection on the principal bundle $N \rightarrow N/H$. The key construction, at least from our point of view, is the one-form α_μ on N , defined by

$$\langle \alpha_\mu(n), v \rangle = \langle \mu, \alpha(n, v) \rangle, \tag{2.19}$$

i.e., $\alpha_\mu = \mu \circ \alpha$. This one-form is used to define what Marsden [33] calls the *shifting map*

$$\text{hor} : T^*N \rightarrow J^{-1}(0) \tag{2.20}$$

$$\beta \rightarrow \beta - \alpha_{J(\beta)}. \tag{2.21}$$

Now α_μ lies in $J^{-1}(\mu)$, so, if we restrict the map hor to $J^{-1}(\mu)$, and quotient by H_μ , we have a map

$$\text{hor}_\mu : (J^{-1}(\mu))/H_\mu \rightarrow J^{-1}(0)/H_\mu \tag{2.22}$$

induced by $p \rightarrow p - \alpha_\mu$. Let J_μ denote the momentum map for $H_\mu \subset H$ (so $J_\mu = J|_{\mathfrak{h}_\mu}$), then $J^{-1}(0)/H_\mu$ embeds in $J_\mu^{-1}(0)/H_\mu \simeq T^*(N/H_\mu)$. Thus the map hor_μ embeds P_μ into $T^*(N/H_\mu)$. The two-form $d\alpha_\mu$ on N drops to a two-form, denoted by β_μ , on the quotient N/H_μ . (This is because α_μ is invariant under the action of H_μ which is the isotropy group of μ .) Let ι denote the embedding of P_μ into $T^*(N/H_\mu)$ via hor_μ , then Kummer [26] has shown that the symplectic form on P_μ is given by

$$\sigma = \iota^* \sigma_0 - \iota^* \pi^* \beta_\mu, \tag{2.23}$$

where $\pi : T^*(N/H_\mu) \rightarrow N/H_\mu$ is the canonical projection and σ_0 is the canonical symplectic form on $T^*(N/H_\mu)$. Note that neither of the two terms on the right-hand side of (2.23) are, in general, symplectic. However, the sum of the two is. Locally, on a coordinate patch $M_A \subset N/H_\mu$, we can write $\sigma = d\Theta_A$. Let $b_A : M_A \rightarrow N$ be a (local) section, then

$$\Theta_A = -\iota^* \theta_0 - \iota^* \pi^* b_A^* \alpha_\mu, \tag{2.24}$$

where θ_0 is the globally defined canonical one-form on $T^*(N/H_\mu)$. Of course, as we noted earlier, it is possible to regard $P_\mu \simeq P_{\mathcal{O}_\mu}$ as a symplectic leaf in the Poisson manifold $(T^*N)/H$. This manifold inherits its Poisson structure from the one on T^*N and a global formula for the Poisson bracket on the reduced space is given in [37]. However, the geometric quantization approach is much more suited to the case when the symplectic form rather than the Poisson bracket is known.

Alternatively, in the Kaluza–Klein picture as generalized by Kerner [22], we could start with a metric on Q and a connection form α on N . As H is compact, a bi-invariant metric exists on H . The metric on N is induced by the connection. To be precise, α defines an orthogonal decomposition $T_n N = V_n \oplus H_n$, $n \in N$, where the horizontal subspace H_n is the kernel of α_n . The metric on $V_n \simeq \mathfrak{h}$ is the one induced from the bi-invariant metric on \mathfrak{h} , whilst the metric on H_n is the pullback of the metric on Q . The metric on N is thus H -invariant since $\rho_{h*} H_n = H_{nh}$ (which is one of the defining properties of a connection). Note that Marsden’s construction of the mechanical connection depends heavily on the given metric on N ; whereas in the Kaluza–Klein picture a given connection is used to construct a metric on N . It is quite straightforward to show that if one starts with this latter case and calculates the mechanical connection then it is merely the connection one started with.

For a particle in a Yang–Mills field, the relevance of the connection with regard to the symplectic leaves of $(T^*N)/H$ is that, as noted by Weinstein [48], until it is chosen there is

no natural projection of $(J^{-1}(\mathcal{O}_\mu))/H$ on T^*Q ; thus the variables conjugate to position on Q are inherently intertwined with the “internal” variables associated with \mathcal{O}_μ . Physically, this means that without a connection we cannot separate the particle’s external momentum from its own internal “position” and “momentum” which is associated with the motion on the coadjoint orbit \mathcal{O}_μ .

2.3. The symplectic leaves of $(T^*N)/H$

Due to the large number of fibre bundles that appear in our discussion, we denote the projection map of a generic bundle C with base space X by $\pi_C \rightarrow X$. For the special case of a cotangent bundle $T^*X \rightarrow X$, we use π_X for the projection map.

To identify the symplectic leaves of $(T^*N)/H$ we use the result, due to Montgomery [36], that $T^*N \simeq N^\# \times \mathfrak{h}^*$, where $N^\#$ denotes the pullback of the bundle N to a bundle over T^*Q using the canonical projection $\pi_Q : T^*Q \rightarrow Q$. The bundle $N^\#$ is represented diagrammatically as

$$\begin{array}{ccc}
 N^\# & \rightarrow & N \\
 \downarrow & & \downarrow \\
 T^*Q & \xrightarrow{\pi_Q} & Q.
 \end{array} \tag{2.25}$$

Further, the momentum map for the action of H , $J : N^\# \times \mathfrak{h}^* \rightarrow \mathfrak{h}^*$, is given by $J(n, \nu) = \nu$. We now briefly review these results.

The first step is that, as noted by Guillemin and Sternberg [13], $N^\#$ has a natural intrinsic realization as $V^0 \subset T^*N$, the annihilator of the vertical bundle $V \subset TN$ ($V_n \subset T_nN$ is the vertical subspace, i.e., it is the subspace tangent to the fibre at $n \in N$). To see this, note that we can write $N^\# = T^*Q \times_Q N = \{(n, p) \in N \times T^*Q : \pi_N \rightarrow_Q(n) = \pi_Q(p)\}$, where $\pi_{N^\# \rightarrow T^*Q}[n, p]_Q = p$ and the projection $\text{pr} : N^\# \rightarrow N$ is given by $\text{pr}[n, p]_Q = n$. We can pull p back to an unique element $\kappa_n = \pi_{N \rightarrow Q}^* p \in T_n^*N$, which is then an element of V_n^0 . This correspondence between $T_{\pi_N \rightarrow_Q(n)}^* Q$ and V_n^0 is clearly bijective.

Just as a connection form, $\alpha_n : T_nN \rightarrow \mathfrak{h}$, defines a unique separation of T_nN into the vertical subspace and horizontal subspace, the dual of the connection form $\alpha_n^* : \mathfrak{h}^* \rightarrow T_n^*N$ defines a unique separation of T^*N into $N^\#$ and \mathfrak{h}^* . Specifically, α induces an H -equivariant isomorphism $\tilde{\alpha} : N^\# \times \mathfrak{h}^* \rightarrow T^*N$ by

$$\tilde{\alpha}(\kappa_n, \nu) = \kappa_n + \alpha_n^* \nu, \tag{2.26}$$

where $\kappa_n \in N^\#$ and we have identified $N^\#$ with $V^0 \subset T^*N$. Recall that H acts on T^*N by cotangent lift; the action of H on $N^\# \times \mathfrak{h}^*$ is the one induced by $\tilde{\alpha}$ and is given by $\rho_h(\kappa_n, \nu) = (\rho_{h^{-1}}^* \kappa_n, \pi_{\text{co}}(h^{-1}) \cdot \nu)$. Note that for $[n, p]_Q \in N^\#$, $\rho_h[n, p]_Q = [nh, p]_Q$. Also $\tilde{\alpha}$ induces a symplectic structure on $N^\# \times \mathfrak{h}^*$ from the canonical one on T^*N .

The moment map $J : N^\# \times \mathfrak{h}^* \rightarrow \mathfrak{h}^*$ for the action of H on $N^\# \times \mathfrak{h}^*$ can be readily computed using the momentum map for the action of H on T^*N given in (2.10). We have, for $X \in \mathfrak{h}^*$, $\langle J(\kappa_n, \nu), X \rangle = \langle \kappa_n, \xi(X) \rangle + \langle \nu, \alpha_n(\xi(X)) \rangle$, where ξ denotes the infinitesimal generator of the right action of H on N . Hence

$$J(\kappa_n, \nu) = \nu. \tag{2.27}$$

Using the results of Section 2.1, we can, using α , immediately identify the symplectic leaves of $(T^*N)/H$ with both P_μ and $P_{\mathcal{O}_\mu}$, where $P_\mu = N^\# / H_\mu$ and $P_{\mathcal{O}_\mu} = N^\# \times_H \mathcal{O}_\mu$.

2.4. A momentum map on the symplectic leaf $P_{\mathcal{O}_\mu}$ – identification of classical observables to be quantized

Recall that in Section 1 we motivated the approach of finding a group \mathcal{G} which acts on the reduced phase space together with a corresponding momentum map. This allows a subclass of observables to be selected which we expect to quantize provided that we can find a suitable representation of \mathcal{G} . Isham [20] has considered the special case where N is a Lie group G (with $H \subset G$) so that $Q = G/H$ is homogeneous. In particular, he considered an action of G on the symplectic leaf $T^*(G/H) \subset (T^*G)/H$ with a corresponding momentum map. We would like to generalize this approach for the present case where G is replaced by the general principal fibre bundle N and we consider any symplectic leaf in $(T^*N)/H$. The guiding principle is that the [left] action of G on the bundle $G \rightarrow G/H = Q$ commutes with the right action of H on G . Hence this action of G determines a subgroup of the group of automorphisms of G . For the general bundle N this group is denoted by $\text{Aut } N$ and consists of all diffeomorphisms ϕ of N which satisfy, for all $h \in H$,

$$\phi(n)h = \phi(nh). \tag{2.28}$$

Note that such a ϕ determines a diffeomorphism of Q , $\bar{\phi} \in \text{Diff } Q$, via

$$\bar{\phi}(\pi(n)) = \pi(\phi(n)), \tag{2.29}$$

where $\pi : N \rightarrow Q$ is the bundle projection. In the general case there is no natural finite-dimensional subgroup of $\text{Aut } N$, thus we are forced to consider the whole group.

We regard the Lie algebra of $\text{Diff } N$ as the set of all complete vector fields on N . Unfortunately the commutator of two vector fields $[A_1, A_2] = -[A_1, A_2]_{\text{LB}}$, where the subscript LB denotes the Lie bracket of the two elements of the Lie algebra. Thus, in order to distinguish between the two brackets we will continue to use this subscript when considering $\text{Diff } N$ (and $\text{Aut } N$).

Drawing on Guillemin and Sternberg’s treatment [14] of the action of the semi-direct product group $\text{Diff } N \ltimes C^\infty(N)$ on T^*N we consider the subgroup $\text{Aut } N \ltimes C^\infty(Q) \subset \text{Diff } N \ltimes C^\infty(N)$. The group law on $\text{Aut } N \ltimes C^\infty(Q)$ being $(\phi_1, f_1) \cdot (\phi_2, f_2) = (\phi_1 \circ \phi_2, f_1 + f_2 \circ \bar{\phi}_1^{-1})$ and the Lie algebra is

$$[(A_1, f_1), (A_2, f_2)]_{\text{LB}} = ([A_1, A_2]_{\text{LB}}, -A_1 f_2 + A_2 f_1). \tag{2.30}$$

Here we have identified $\mathcal{L}(C^\infty(Q))$ with $C^\infty(Q)$. Now $\text{Aut } N \ltimes C^\infty(Q)$ acts symplectically on T^*N (because $\text{Diff } N \ltimes C^\infty(N)$ does) and the action is given by

$$\tau_{(\phi, f)}\beta_n = \phi^{-1*}(\beta_n) - (d\pi^* f)_{\phi(n)}, \quad \beta_n \in T_n^*N. \tag{2.31}$$

This choice of the group $\text{Aut } N \ltimes C^\infty(Q)$, whose action on T^*N we are interested in, agrees with that of Landsman [30].

Guillemin and Sternberg [14] give the equivariant momentum map, $J : T^*N \rightarrow \mathcal{L}(\text{Aut } N \ltimes C^\infty(Q))^*$, for this action as

$$\langle J(p_n), (A, f) \rangle = \langle p_n, A \rangle + \pi^* f(n). \tag{2.32}$$

Now suppose $J(p'_n) = J(p_n)$. Clearly $\pi(n') = \pi(n)$ and hence $n' = nh$ for some $h \in H$. We thus have $\langle p'_{nh}, A_{nh} \rangle = \langle p_n, A_n \rangle$. But $\mathcal{L}(\text{Aut } N)$ consists of all smooth vector fields on N that are H -invariant (e.g., see [14]), i.e., they satisfy $\rho_{h*}(A_n) = A_{nh}$, for the flow of such a vector field consists of transformations belonging to $\text{Aut } N$. Hence $\rho_h^* p'_{nh} = p_n$. Thus the fibres of J are generated by the right action of H . Further the action τ defined in (2.31) commutes with the right action of H ; this explains the reasoning behind choosing $\pi^* C^\infty(Q)$ rather than $C^\infty(N)$. Thus τ drops to an action $\bar{\tau}$ on $(T^*N)/H$.

In passing, we note that there is a relation between the momentum map J and the dual pairs of Section 2.1. Specifically, let J_R be the momentum map for the right action of H as given in (2.18). We know from Section 2.1 that we have the dual pair

$$\mathfrak{h}^* \xleftarrow{J_R} T^*N \xrightarrow{\pi} (T^*N)/H. \tag{2.33}$$

Now note that $J(T^*N)$ is finite dimensional. Further, using (2.32), we can identify $J(T^*N)$ with $M = \{(\beta, q) \in T^*N \times Q : \pi_{T^*N \rightarrow Q}(\beta) = q\}$ where $q \in Q$ is regarded as an element of $\mathcal{L}(C^\infty(Q))^*$ via $\langle q, f \rangle = f(q)$ for $f \in \mathcal{L}(C^\infty(Q)) \simeq C^\infty(Q)$. The elements of the space $\pi^* C^\infty((T^*N)/H)$ of functions on T^*N are constant on the fibres of J and hence this space coincides with the space $J^* C^\infty(M)$. Thus, we have the full dual pair

$$\mathfrak{h}^* \xleftarrow{J_R} T^*N \xrightarrow{J} J(T^*N) \subset \mathcal{L}(\text{Aut } N \ltimes C^\infty(Q))^*. \tag{2.34}$$

We then note that J induces a symplectic diffeomorphism \bar{J}_μ which maps the symplectic leaf $P_{\mathcal{O}_\mu} = (J_R^{-1}(\mathcal{O}_\mu))/H \subset (T^*N)/H$ to a symplectic leaf in $J(T^*N) \subset \mathcal{L}(\text{Aut } N \ltimes C^\infty(Q))^*$. Furthermore, the map \bar{J}_μ is a momentum map for the action $\bar{\tau}$ on $(T^*N)/H$.

We have thus achieved our goal and we can now write down the classical observables we expect to be able to quantize. These are given by $\{\hat{J}_\mu(A, f) : A \in \mathcal{L}(\text{Aut } N), f \in C^\infty(Q)\}$ where $\hat{J}_\mu : \mathcal{L}(\text{Aut } N \ltimes C^\infty(Q)) \rightarrow C^\infty(P_{\mathcal{O}_\mu})$ is given by $(\hat{J}_\mu(A, f))[p_n] = \langle \bar{J}_\mu[p_n], (A, f) \rangle$ with $[p_n] \in P_{\mathcal{O}_\mu}$, cf. (2.3). Recalling that $P_{\mathcal{O}_\mu} = N^\# \times_H \mathcal{O}_\mu$, we then have for $p_n = (\beta_n, v) \in N^\# \times \mathcal{O}_\mu$, using (2.26)

$$\langle \bar{J}_\mu[p_n], (A, f) \rangle = \langle \beta_n + \alpha_n^* v, A \rangle + \pi^* f(n). \tag{2.35}$$

Let s be a local section of the bundle $N \rightarrow Q$. This allows us to choose a specific element in each of the equivalence classes $N^\# \times_H \mathcal{O}_\mu$, so that

$$(\hat{J}_\mu(A, f))[\beta_{s(q)}, v]_H = \langle \beta_{s(q)}, A \rangle + \langle v, \alpha_{s(q)}(A) \rangle + f(q). \tag{2.36}$$

This expression simplifies if we use local coordinates. Now $N^\# / H = T^*Q$, and locally $P_{\mathcal{O}_\mu}$ is like $(N^\# / H) \times \mathcal{O}_\mu$. Thus, let $(h^1, \dots, h^{d_H}, q^{d_H+1}, \dots, q^{d_N})$ be local coordinates on

N , where $(q^{d_H+1}, \dots, q^{d_N})$ are coordinates on Q and (h^1, \dots, h^{d_H}) are coordinates on the fibre H . Let $p_{d_H+1}, \dots, p_{d_N}$ be the corresponding components of covectors on T^*Q . Then we can label a point $[\beta_{s(q)}, v]_H$ in $N^\# \times_H \mathcal{O}_\mu$ by $(q^{d_H+1}, \dots, q^{d_N}, p_{d_H+1}, \dots, p_{d_N}, v)$. So we can write $\alpha_{s(q)}(A) = X(q^{d_H+1}, \dots, q^{d_N})$ where $X \in \mathfrak{h}$, together with $(\pi_{N \rightarrow Q^*} A)_q = v^\gamma (q^{d_H+1}, \dots, q^{d_N}) (\partial/\partial q^\gamma)$ where $\gamma = d_H + 1, \dots, d_N$. Thus, setting $\beta_{s(q)} = \pi_{N \rightarrow Q}^* p$ with $p = p_\gamma dq^\gamma$, we have

$$\hat{J}_\mu(A, f)[\beta_{s(q)}, v]_H = v^\gamma (q^{d_H+1}, \dots, q^{d_N}) p_\gamma + \langle v, X(q^{d_H+1}, \dots, q^{d_N}) \rangle + f(q^{d_H+1}, \dots, q^{d_N}). \tag{2.37}$$

This gives the classical observables which we expect to quantize.

2.4.1. The reduced Hamiltonian

The Hamiltonian H_0 on T^*N drops to a reduced Hamiltonian on the symplectic leaves of $(T^*N)/H$. In particular, when a symplectic leaf is identified with $\mathcal{P}_{\mathcal{O}_\mu}$, the reduced Hamiltonian $H_{\mathcal{O}_\mu}$ is given by [33]

$$H_{\mathcal{O}_\mu}(q, p, v) = \frac{1}{2} \|p\|^2 + \frac{1}{2} \langle v, \mathbb{I}(s(q))v \rangle + V(s(q)), \tag{2.38}$$

where (q, p, v) labels locally a point in $N^\# \times_H \mathcal{O}_\mu$ as above. Denote by A_γ the element of $\mathcal{L}(\text{Aut } N)$ such that $\pi_{N \rightarrow Q^*} A_\gamma = (\partial/\partial q^\gamma)$ and $\alpha(A_\gamma) = 0$. Then $\hat{J}_\mu(A_\gamma, 0)[\beta_{s(q)}, v]_H = p_\gamma$. Similarly denote by A_I the element of $\mathcal{L}(\text{Aut } N)$ such that $\pi_{N \rightarrow Q^*} A_I = 0$ and $\alpha(A_I) = T_I$ where $\{T_J : J = 1, \dots, d_H\}$ is a basis for \mathfrak{h} . Hence $\hat{J}_\mu(A_I, 0)[\beta_{s(q)}, v]_H = \langle v, T_I \rangle$. We can then write the reduced Hamiltonian as

$$H_{\mathcal{O}_\mu} = \frac{1}{2} \mathfrak{g}^{\alpha\beta} \hat{J}_\mu(A_\alpha, 0) \hat{J}_\mu(A_\beta, 0) + \frac{1}{2} \mathbb{I}^{IJ} \hat{J}_\mu(A_I, 0) \hat{J}_\mu(A_J, 0) + \hat{J}_\mu(0, V_0). \tag{2.39}$$

Here $V_0 \in C^\infty(Q)$ is such that $\pi_{N \rightarrow Q}^* V_0 = V$ while $\{\mathfrak{g}_{\alpha\beta}\}$ and $\{\mathbb{I}_{IJ}\}$ are the metrics on Q and \mathfrak{h} respectively ($\mathfrak{g}_{\alpha\beta} \mathfrak{g}^{\beta\gamma} = \delta_\alpha^\gamma$, $\mathbb{I}^{IK} \mathbb{I}_{KJ} = \delta^I_J$).

3. Geometric quantization

We give a brief outline of the main procedures of geometric quantization. The reader is referred to Woodhouse [50], Sniatycki [45] or Puta [40] for comprehensive expositions.

3.1. Prequantization and polarizations

Prequantization is the process of finding the Hilbert space \mathcal{H} described in Section 1 together with the map $f \rightarrow \hat{f}$ which links classical observables with their counterpart quantum operators. A complex Hermitian line bundle B over the symplectic space M is introduced along with a connection ∇ on B with curvature $\hbar^{-1} \sigma$, where σ is the symplectic form on M . The bundle B is called the *prequantum line bundle*. An inner product $\langle \cdot, \cdot \rangle$ on $\Gamma(B)$ (the sections of B) is given by

$$\langle s_1, s_2 \rangle = \int_M (s_1, s_2) \sigma^n, \quad (3.1)$$

where $\dim M = 2n$. We restrict \mathcal{H} to be the space of square-integrable sections. Each observable, $f \in C^\infty(M)$, corresponds to the operator \hat{f} , where

$$\hat{f}s = -i\hbar \nabla_{\xi_f} s + fs, \quad (3.2)$$

and ξ_f is the Hamiltonian vector field associated to f . This ensures that conditions (Qi)–(Qiii) of Section 1 hold. If ξ_f is complete then \hat{f} is essentially self-adjoint (on a suitable domain).

Associated to each observable f is a vector field V_f on B characterized by

$$\begin{aligned} \pi_{B \rightarrow M} V_f &= \xi_f; \\ \hbar \langle \tilde{\Theta}, V_f \rangle &= \hbar \langle \tilde{\Theta}, V_f \rangle = -f \circ \pi_{B \rightarrow M}. \end{aligned} \quad (3.3)$$

Here $\tilde{\Theta}$ is the connection one-form on the prequantum bundle B and $\bar{\tilde{\Theta}}$ is its complex conjugate. Let ρ_t denote the flow of ξ_f , and δ_t the flow of V_f . For a section $s \in \Gamma(B)$ a linear “pullback” action $\hat{\rho}_t : \Gamma(B) \rightarrow \Gamma(B)$ can be defined by

$$\delta_t(\hat{\rho}_t s(m)) = s(\rho_t(m)). \quad (3.4)$$

Then, $\hat{\rho}_t$ is related to the quantum operator \hat{f} via

$$\left. \frac{d\hat{\rho}_t}{dt} \right|_{t=0} = i\hbar^{-1} \hat{f}. \quad (3.5)$$

For a given symplectic manifold, a prequantum bundle does not always exist. This leads to what are called quantization or integrality conditions which determine if and when a prequantum bundle exists. Such conditions are usually formulated as a requirement on an integral of the symplectic form or in terms of de Rham cohomology classes.

The next step in geometric quantization, once the prequantum bundle has been found, is to construct a *polarization* of the symplectic manifold. Then the Hilbert space \mathcal{H} is replaced by sections which are parallel along the polarization. Such sections are called polarized sections. The class of observables that can be quantized is then restricted to those for which the flow of the corresponding Hamiltonian vector field preserves the polarization. When the polarization contains real directions the step of restricting attention to the polarized sections necessitates a change in the inner product on \mathcal{H} and a corresponding alteration of the operator \tilde{f} corresponding to an observable f . Our approach, as will be seen in Section 4.3, is to utilize the well known approach of using half-densities (e.g. see Simms and Woodhouse [43] or Puta [40]). We have avoided using the so-called half-form correction to geometric quantization (e.g. see Blattner [5,50,40]) because, firstly, for the coadjoint orbits of compact H there is no problem in constructing an inner product for the quantization and also the metaplectic correction arising out of half-form quantization has the somewhat perverse effect of shifting the value of the highest weight associated with a representation. Secondly, for the reduced cotangent bundle, there is a very natural inner product arising by simply integrating over

N/H , and also, as we were only concerned with quantizing functions linear in momentum (i.e., the generators of the canonical group), we are not interested in pairing wave functions arising from different polarizations.

3.2. Quantization on coadjoint orbits

Much has been written on the subject of geometric quantization on coadjoint orbits; e.g., see Kostant [25], Woodhouse [50] and Baston and Eastwood [4]. We briefly outline the main steps.

Let H be a compact connected Lie group with Lie algebra \mathfrak{h} ; let $\mu \in \mathfrak{h}^*$ and let $\mathcal{O}_\mu \subset \mathfrak{h}^*$ denote the coadjoint orbit of μ . Recall, that in Section 2.1, we saw that we could regard H as a presymplectic manifold. Also, we noted that for compact H (assumed here) H_μ , the isotropy group of μ , is connected; then the reduction of H by the left action of H_μ gives $H_\mu \backslash H \simeq \mathcal{O}_\mu \subset \mathfrak{h}^*$. The symplectic 2-form $\omega_{\mathcal{O}^-}$ on $H_\mu \backslash H$ is given by $\pi^* \omega_{\mathcal{O}^-} = \omega_\mu$ where $\pi : H \rightarrow H_\mu \backslash H$ is the projection and ω_μ is the restriction of the canonical 2-form on T^*H to $H \times \{\mu\} \simeq H$ where we are working with the right trivialization of T^*H . Thus, from (2.13), $\omega_\mu = -d\theta_\mu$ where $\theta_\mu(h) = \rho_{h^{-1}}^* \mu$.

Having detailed the symplectic manifold $(H_\mu \backslash H, \omega_{\mathcal{O}^-})$, the next step is to construct the prequantum bundle. Drawing on Woodhouse [50], Kostant’s formulation of the integrality condition on $\omega_{\mathcal{O}^-}$ can be expressed as the requirement that $-i\hbar^{-1}\mu$ should be the gradient at e of a homomorphism $\chi_\mu : H_\mu \rightarrow \mathbb{T}$, where \mathbb{T} is the circle group.

The prequantum line bundle B is given by $B = H \times_{H_\mu} \mathbb{C}$, i.e., $H \times \mathbb{C}$ modulo the equivalence relation $(h, z) \sim (h_\mu h, \chi_\mu(h_\mu)z)$ for $h \in H, h_\mu \in H_\mu$. This bundle has a connection whose curvature is $\hbar^{-1}\omega_{\mathcal{O}^-}$. The connection can be either considered in the light of [50] or in the following manner. The principal bundle $H \rightarrow H_\mu \backslash H$ has the canonical H -invariant (under right action) connection (e.g., see [23]); by the assumption on the integrality of μ , there is a representation of H_μ into $U(1)$. Under the derivative of this representation, the canonical connection becomes a connection on B' with curvature $\hbar^{-1}\omega_{\mathcal{O}^-}$.

We can identify the sections of B with functions $\phi : H \rightarrow \mathbb{C}$ satisfying

$$\phi(h_\mu h) = \chi_\mu(h_\mu)\phi(h). \tag{3.6}$$

There is an induced representation, π_μ of H , on these functions defined by

$$(\pi_\mu(h')\phi)(h) = \phi(hh'). \tag{3.7}$$

An inner product is given by

$$\langle \phi_1, \phi_2 \rangle = \int_{H_\mu \backslash H} d([h]_{H_\mu}) \langle \phi_1(h), \phi_2(h) \rangle_{\mathbb{C}}. \tag{3.8}$$

We only require two more facts concerning the quantization on coadjoint orbits. The first of these is that there is a positive H -invariant polarization on $H_\mu \backslash H$. This is a standard result, e.g. see Woodhouse [50, pp. 102–105]. The term H -invariance means that for each

$[s] \in H_\mu \backslash H$ we have $\rho_{h^*} P_{[s]}^\mathcal{O} = P_{[s]h}^\mathcal{O}$, where H acts naturally on the right of $H_\mu \backslash H$. The existence of a positive Kähler polarization means that we can take the Hilbert space $\mathcal{H}_\mathcal{O}$ for the quantization of \mathcal{O}_μ to be the space of square-integrable sections of the prequantum line bundle $B \rightarrow \mathcal{O}_\mu$. Note also the fact that H is compact means that $\mathcal{H}_\mathcal{O}$ is finite-dimensional.

The second fact we require is that the representation π_μ of H acting on the polarized sections of B is irreducible. This has been proved by Kostant [25]. The techniques involved (which centre around the use of the Borel–Weil theorem) do not play any part in the remainder of the paper and we will omit the details. It suffices to mention that by choosing an appropriate value of μ , all finite irreducible unitary representations of H are obtained in this way. (See Woodhouse [50, pp. 176–177] for a discussion on the value of μ which generates a given representation.)

4. Geometric quantization of the symplectic leaves of $(T^*N)/H$

Recall that the reduced phase space of our constrained mechanical system can be identified with a symplectic leaf of $(T^*N)/H$. We now apply the technique of geometric quantization to these symplectic spaces.

4.1. The prequantum line bundle $B \rightarrow P_\mu$

We require a (complex) hermitian line bundle $B \rightarrow P_\mu$ and a connection ∇ on B with curvature $\hbar^{-1}\sigma$, with σ given in (2.23). In particular, we saw in Section 2.2 that the symplectic form on P_μ was built from the 2-form β_μ defined on N/H_μ and the canonical 2-form on $T^*(N/H_\mu)$. Thus, we aim to find a line bundle B' over N/H_μ and a connection on B' with curvature $-\hbar^{-1}\beta_\mu$. We can pullback B' by π to form $\pi^*B' \rightarrow T^*(N/H_\mu)$, where π is the same as in (2.23). The tensor product bundle formed from π^*B' and the trivial bundle $B_0 = T^*(N/H_\mu) \times \mathbb{C}$ will yield a line bundle, $B_1 = \pi^*B' \otimes B_0$, with curvature the sum of the curvatures of π^*B' and B_0 . (The simple expression for the curvature is a consequence of the additivity of the Chern character under the formation of tensor product bundles.) Now B_0 admits a connection with curvature $\hbar^{-1}\sigma_0$, thus, by considering (2.23), $B = \iota^*B_1$ will be a line bundle over P_μ with the desired connection, where ι is defined just before (2.23).

The key point in constructing the line bundle $B' \rightarrow N/H_\mu$ is that α defines a connection α' on $N \rightarrow N/H_\mu$ via $\alpha' = \text{pr} \circ \alpha$ where $\text{pr} : \mathbf{h} \rightarrow \mathbf{h}_\mu$ is the projection relative to the metric on H . A representation χ_μ of H_μ into $U(1)$ then allows us to define the associated line bundle $B' = N \times_{H_\mu} \mathbb{C}$ (where $(n, z) \sim (nh_\mu, \chi_\mu(h_\mu^{-1})z)$ for $h_\mu \in H_\mu$) with a corresponding connection. Of course there is a restriction on χ_μ if B' is to have the desired connection. Interestingly, the condition on χ_μ is the same as Kostant's formulation of the integrality condition for the quantization of coadjoint orbits described in Section 3.2, i.e., $-i\hbar^{-1}\mu$ should be the gradient at e of a homomorphism $\chi_\mu : H_\mu \rightarrow \mathbb{T}$ where \mathbb{T} is the circle group. To see this, note that χ_μ defines a representation of H_μ into $U(1)$ and its derivative defines a representation $\chi'_\mu : \mathbf{h}_\mu \rightarrow \mathbb{C}$ which is given by $\chi'_\mu(A) = -i\hbar^{-1}\langle \mu, A \rangle$. Under this

derivative the connection α' gives a connection on the associated bundle B' with curvature $-\hbar^{-1}\beta_\mu$, where β_μ denotes the two-form $d\alpha_\mu$ dropped to N/H_μ . Specifically, let \mathcal{A} be the local expression for α' , then from the definition of a covariant derivative

$$\nabla_X(\psi s) = (X(\psi) + \chi'_\mu(\mathcal{A}(X))\psi)s, \quad X \in \Gamma(T(N/H_\mu)). \tag{4.1}$$

Here, s denotes the unit section and ψ is a complex valued function. But $\chi'_\mu(\mathcal{A}(X)) = -i\hbar^{-1}\langle \mathcal{A}_\mu, X \rangle$ where $\mathcal{A}_\mu = \langle \mu, \mathcal{A} \rangle$. Hence

$$\nabla_X(\psi s) = (X(\psi) - i\hbar^{-1}\langle \mathcal{A}_\mu, X \rangle\psi)s, \tag{4.2}$$

so \mathcal{A}_μ determines a connection with curvature $-\hbar^{-1}d\mathcal{A}_\mu = -\hbar^{-1}\beta_\mu$. For future reference we note that we can identify sections of $B' \rightarrow N/H_\mu$ with functions $\gamma : N \rightarrow \mathbb{C}$ such that

$$\gamma(nh_\mu) = \chi_\mu(h_\mu^{-1})\gamma(n) \quad \forall h_\mu \in H_\mu. \tag{4.3}$$

For completeness we relate our approach to that of Woodhouse [50, Proposition 8.4.9] for the construction of the prequantum line bundle for the reduction of a symplectic manifold. Specifically, in our present notation, Woodhouse defines the line bundle to be $N \times \mathbb{C}$ quotiented by the equivalence relation $(n_1, z_1) \sim (n_2, z_2)$ if $\pi(n_1) = \pi(n_2)$ and $z_2 = z_1 \exp(-i\hbar^{-1} \int_{n_1}^{n_2} \alpha_\mu)$. Here $\pi : N \rightarrow N/H_\mu$ is the projection map and the precise path of the integral does not matter since it is supposed that α_μ satisfies the integrality condition $(1/2\pi\hbar) \int_\gamma \alpha_\mu \in \mathbb{Z}$ whenever γ is a closed curve in a fibre of $N \rightarrow N/H_\mu$. (Note that Woodhouse's construction does not require the 2-form $d\alpha_\mu$ to be symplectic.)

From the defining properties of a connection, it immediately follows that, for $A \in \mathfrak{h}_\mu$, $\langle \alpha_\mu, \xi(A) \rangle = \langle \mu, A \rangle$ and, additionally using the H_μ invariance of μ , $\rho_{h_\mu}^* \alpha_\mu = \alpha_\mu$, where $h_\mu \in H_\mu$. Considering the equivalence relation defined above, clearly $n_2 = n_1 h_\mu$ for some $h_\mu \in H_\mu$. Thus, for $h_\mu = e^A$, where $A \in \mathfrak{h}_\mu$, we have

$$\exp\left(\frac{-i}{\hbar} \int_{n_1}^{n_1 e^A} \alpha_\mu\right) = \exp\left(\frac{-i}{\hbar} \langle \mu, A \rangle\right). \tag{4.4}$$

Recall that H_μ is connected (see Section 2.1); hence we can define $\chi_\mu : H_\mu \rightarrow \mathbb{C}$ by

$$\chi_\mu(h_\mu) = \exp\left(\frac{-i}{\hbar} \int_n^{nh_\mu} \alpha_\mu\right). \tag{4.5}$$

Note that the right-hand side of (4.5) is independent of n and so χ_μ is well defined. Thus, the integrality condition is equivalent to χ_μ being a single valued function on H_μ . Further,

$$\chi_\mu(h_\mu e^{tA}) = \chi_\mu(h_\mu)\chi_\mu(e^{tA}), \tag{4.6}$$

and hence that $\chi_\mu(h_\mu h'_\mu) = \chi_\mu(h_\mu)\chi_\mu(h'_\mu)$ for all $h'_\mu, h_\mu \in H_\mu$. Now, by noting that $\langle d\chi_\mu, \xi(A) \rangle = -i\hbar^{-1}\chi_\mu(h_\mu)\langle \mu, A \rangle$, we see that χ_μ is a homomorphism $\chi_\mu : H_\mu \rightarrow \mathbb{T}$ whose gradient at e is $-i\hbar^{-1}\mu$. Reversing the argument, it can be seen that the converse holds. Thus we find the same condition on χ_μ as before.

Note that, given α , our construction defines a unique line bundle (B', ∇) . This is in contrast with the usual situation in geometric quantization because there the symplectic 2-form σ is the starting point and this does not define a unique one-form θ such that $d\theta = \sigma$; the construction of the line bundle uses the 1-form θ , and thus this process does not, in general, give a unique bundle unless the symplectic space is simply connected (e.g., see [50]). Our approach avoids this problem because we start with α_μ rather than β_μ .

Recalling the comments made in the opening paragraph of this section we have now proved:

Theorem 4.1. *Let B_0 be the trivial bundle $T^*(N/H_\mu) \times \mathbb{C}$ with a connection determined by the local connection form $-\hbar^{-1}\theta_0$, where θ_0 is the globally defined canonical one-form on $T^*(N/H_\mu)$. Then the prequantum line bundle $B \rightarrow P_\mu$ is given by $i^*(\pi^*B' \otimes B_0)$ where $\pi : T^*(N/H_\mu) \rightarrow N/H_\mu$ is the canonical projection and $B' = N \times_{H_\mu} \mathbb{C}$ is the line bundle given above.*

For clarity and for future reference we note that the bundle B has local connection one-forms $-i\hbar^{-1}(i^*\theta_0 + i^*\pi^*\gamma^*\alpha_\mu)$ on $\pi^{-1}(M)$, $M \subset N/H_\mu$, where $\gamma : M \rightarrow N$ is a local section.

If we use principal bundles rather than their associated vector bundles, the trivialization $T^*N \simeq N^\# \times \mathfrak{h}^*$ allows more explicit forms for the various bundles just described to be given. The key point is to realize that the map hor defined in (2.21) is no more than the shift $J^{-1}(\mu) \simeq N^\# \times \{\mu\} \rightarrow N^\#$. Also, by considering $\pi_{N/H_\mu}^* N$ to be the annihilator of the vertical bundle of $N \rightarrow N/H_\mu$, we see that $\pi_{N/H_\mu}^* N \simeq N^\# \times \mathfrak{n}^*$, where $\mathfrak{n}^* \subset \mathfrak{h}^*$ is defined to be the annihilator of $\mathfrak{h}_\mu \subset \mathfrak{h}$. Then, the pullback bundles of $N \rightarrow N/H_\mu$ are given by the following diagram.

$$\begin{array}{ccccc}
 i^*\pi_{N/H_\mu}^* N \simeq N^\# & \rightarrow & \pi_{N/H_\mu}^* N \simeq N^\# \times \mathfrak{n}^* & \rightarrow & N \\
 \downarrow & & \downarrow & & \downarrow \\
 P_\mu \simeq N^\#/H_\mu & \xrightarrow{i} & T^*(N/H_\mu) \simeq N^\# \times_{H_\mu} \mathfrak{n}^* & \xrightarrow{\pi_{N/H_\mu}} & N/H_\mu
 \end{array} \tag{4.7}$$

The bundle $i^*\pi_{N/H_\mu}^* B'$ is given by $N^\# \times_{H_\mu} \mathbb{C}$. (The prequantum bundle B has the same structure but the connection is not the one induced from B' .)

4.1.1. The Aharanov–Bohm effect

Briefly [2,50,39], Aharanov and Bohm considered the case of a particle with charge e moving in the region outside an [infinitely] long cylinder, so that the configuration space Q of the system is no longer simply connected. Inside the cylinder there is a nonvanishing magnetic field; even if the magnetic field in Q vanishes, there is no gauge in which the magnetic vector potential \mathcal{A} vanishes in Q . It is found that the potential influences the motion of the particle, in that the phase change of the wave function of the particle around a closed loop surrounding the cylinder is not zero, but is given by

$$\exp\left(\frac{ie}{\hbar} \oint \mathcal{A}_a dq^a\right). \tag{4.8}$$

The Aharanov–Bohm effect in the context of geometric quantization is well understood [17,16]. We now quickly show how our approach reproduces the expected results.

The phenomena of electromagnetic fields is described by a $U(1)$ gauge theory. Thus we have $H = U(1)$ and hence $H_\mu = U(1)$ also. A magnetic vector potential \mathcal{A} corresponds to a connection α on the bundle $N \rightarrow Q = N/H$ (i.e., \mathcal{A} is the local form on Q for α). Let $\gamma : [0, 1] \rightarrow N/H_\mu = N/H$ be a closed loop. Denote by $\tilde{\gamma}$ the horizontal lift of γ to N with respect to the connection α' . (Note that $\alpha' = \alpha$ here since $H_\mu = H$.) Define $h_\mu \in H_\mu$ by

$$\tilde{\gamma}(1) = \tilde{\gamma}(0)h_\mu, \tag{4.9}$$

then we have, by a direct consequence of the construction of $\tilde{\gamma}$,

$$h_\mu = \exp\left(-\oint \mathcal{A}_a dq^a\right). \tag{4.10}$$

Now the phase change in ψ , a section of the bundle B' , on going round the loop γ by parallel transport is just $\chi_\mu(h_\mu)$. From (4.5) and (4.4), we have immediately

$$\chi_\mu(h_\mu) = \exp\left(\frac{i\mu}{\hbar} \oint \mathcal{A}_a dq^a\right), \tag{4.11}$$

where we have identified $\mathcal{L}(U(1))$ with \mathbb{R} . This agrees with (4.8) since $\mathcal{O}_\mu = \{\mu\} \in \mathfrak{h}^*$ is identified with the charge e of the particle. Hence our construction *automatically* gives the physically correct choice of the prequantum line bundle. At this stage it is not clear how this generalizes to non-Abelian gauge groups. We will return to this at the end of Section 4.4.

4.2. A polarization for P_μ

Recall that we may consider $P_\mu \simeq P_{\mathcal{O}_\mu} = N^\# \times_H \mathcal{O}_\mu$. Now, we saw that, in Section 3.2, the coadjoint orbit $\mathcal{O}_\mu \simeq H_\mu \backslash H$ has a natural H -invariant positive Kähler polarization $P^\mathcal{O}$. (The H -invariance of the polarization means that for each $[s] \in H_\mu \backslash H$, we have $\rho_{h^*} P^\mathcal{O}_{[s]} = P^\mathcal{O}_{[s]h}$, where H acts naturally on the right of $H_\mu \backslash H$.)

For a cotangent bundle $T^*Q \rightarrow Q$, a natural polarization is given by the complexified vertical subspace at each $u \in T^*Q$ [50], i.e., the complexified subspace of $T_u(T^*Q)$ which is tangent to the fibre. (This is called the vertical polarization.) In a similar manner, we may define an integrable complex distribution P_0 (a sub-bundle of the complexified tangent bundle) on the bundle $N^\# \rightarrow N$ simply by taking the complexified subspace of the tangent space which is tangent to the fibre. The fibre of the bundle $N^\# \rightarrow N$ at $n \in N$ corresponds to V_n^0 , the annihilator of the vertical subspace V_n . Now $\rho_{h^*} V_n = V_{nh}$, so that $\rho_{h^{-1}}^* V_n^0 = V_{nh}^0$. Hence, the distribution P_0 is invariant under the right action of H .

The direct sum of the polarization $P^\mathcal{O}$ on \mathcal{O}_μ and the distribution P_0 on $N^\#$ gives a new H -invariant distribution P' on $N^\# \times \mathcal{O}_\mu$. The projection of P' onto $N^\# \times_H \mathcal{O}_\mu$ is the distribution given by

$$P_m = \text{pr}_* P'_u, \tag{4.11}$$

where $\text{pr} : N^\# \times \mathcal{O}_\mu \rightarrow N^\# \times_H \mathcal{O}_\mu$ is the projective map and u is any element of $\text{pr}^{-1}(m)$. The precise choice of u is irrelevant since both $P^\mathcal{O}$ and P_0 are invariant under the action of H . For suppose $u_1, u_2 \in \text{pr}^{-1}(m)$ then $u_1 = u_2 \cdot h$ for some $h \in H$. By H -invariance, $\text{pr}_* P'_{u_1} = \text{pr}_* \rho_{h*} P'_{u_2} = \text{pr}_* P'_{u_2}$ as required.

Theorem 4.2. *The distribution P is a polarization for $P_{\mathcal{O}_\mu}$.*

Proof. Firstly, P and $P + \bar{P}$ are involutory since the push-forward map pr_* preserves commutators. Further, by construction we see that P is smooth and also $P_m \cap \bar{P}_m = \text{pr}_*(P_0 \oplus \{0\})$. Thus $P_m \cap \bar{P}_m$ is of constant dimension. We finally need to show that P is maximally isotropic. Clearly P has the right dimension since $\dim P = \dim N - \dim \mathfrak{h}^* + \frac{1}{2} \dim \mathcal{O}_\mu$ whilst $\dim N^\# \times_H \mathcal{O}_\mu = 2 \dim N - \dim \mathfrak{h}^* - \dim H + \dim \mathcal{O}_\mu$. To show that P is isotropic we need the expression for the symplectic form on $P_{\mathcal{O}_\mu}$ given in (2.7). Since we are using the trivialization $T^*N \simeq N^\# \times \mathfrak{h}^*$, Ω should be replaced by $\tilde{\alpha}^* \Omega$ which is the induced symplectic form on $N^\# \times \mathfrak{h}^* \simeq T^*N$. We have

$$\Omega_{\mathcal{O}}(P_m, P_m) = (j_{\mathcal{O}}^* \tilde{\alpha}^* \Omega)(P'_u, P'_u) - (J_{\mathcal{O}}^* \omega_{\mathcal{O}})(P'_u, P'_u) \tag{4.12}$$

$$= 0, \tag{4.13}$$

where to justify $\Omega(\tilde{\alpha}_* j_{\mathcal{O}*} P'_u, \tilde{\alpha}_* j_{\mathcal{O}*} P'_u) = 0$, we consider local canonical coordinates $\{q^i, p_j\}$ on T^*N . Then, we see that $\tilde{\alpha}_* j_{\mathcal{O}*} P'_u$ is spanned by $\{\partial/\partial p_j\}$, and thus the desired result. Hence P is maximally isotropic and is a polarization for $P_{\mathcal{O}_\mu} \simeq P_\mu$. \square

4.3. Quantization

Having found a polarization for P_μ , the standard approach in geometric quantization is to replace the pre-Hilbert space of smooth square-integrable sections of the prequantum line bundle B with the subspace of square-integrable polarized sections of B . The quantum operator corresponding to a classical observable is defined on the polarized sections of the prequantum line bundle B . However, these sections are not square-integrable on P_μ . Thus, in a manner analogous to that described in [50], we alter the quantization process so that we integrate over Q rather than P_μ .

Briefly, let $\pi : P_\mu \rightarrow Q$ be the canonical projection and let $\Delta_Q \rightarrow Q$ denote the line bundle $\Lambda^n T^*_\mathbb{C} Q$. (Here $\Lambda^n V$ is the n -fold exterior power of a vector space V and $n = \dim Q$.) Then, define $K_D = \pi^* \Delta_Q \subset \Lambda^n T^*_\mathbb{C} P_\mu$. The bundle Δ_Q is trivial so K_D is too. Thus, we can define $\delta_D = \sqrt{K_D}$. We now replace B by $B_K = B \otimes \delta_D$ and consider polarized sections of this bundle. In terms of the bundle E , we can view sections of the new bundle B_K as sections of the bundle $E \otimes \sqrt{\Delta_Q} \rightarrow Q$. Sections of this bundle are of the form $\tilde{s} = s\eta$ where $s \in \Gamma(E)$ and $\eta \in \Gamma(\sqrt{\Delta_Q})$. The inner product for such sections is

$$(\tilde{s}_1, \tilde{s}_2) = \int_Q \langle s_1(q), s_2(q) \rangle_{\mathcal{H}_\mu}(\eta_1, \eta_2), \tag{4.14}$$

where $(\eta_1, \eta_2) = \eta_1 \bar{\eta}_2 \in \Delta_Q$.

The quantum operator \tilde{f} corresponding to a classical observable f is given by [50]

$$\tilde{f}\tilde{s} = \hat{f}(s)v - \frac{1}{2}i\hbar s(\operatorname{div} \xi_f)\eta, \tag{4.15}$$

where $\tilde{s} = s\eta$ and div is defined with respect to η via $\tilde{\mathcal{L}}_V\eta^2 = (\operatorname{div} V)\eta^2$. (Here, $\tilde{\mathcal{L}}$ denotes the Lie derivative.) However, only certain observables can be quantized. Specifically, the flow of ξ_f must preserve the polarization and, additionally, we are interested in the case when ξ_f is complete so that the operator \tilde{f} is essentially self-adjoint (on a suitable domain). For a vertical polarization of T^*Q , the form of such an observable is

$$f = v^i(q)p_i + u(q), \tag{4.16}$$

where $v \in \Gamma(TQ)$ and $u \in C^\infty(Q)$ [50]. For a Kähler polarization, ξ_f must be a Killing vector [50]. We now consider a special case of the latter, namely, a Kähler polarization on a coadjoint orbit, $\mathcal{O}_\mu \simeq H_\mu \setminus H$. We can regard $X \in \mathfrak{h}$ as an element of $C^\infty(\mathcal{O}_\mu) \subset C^\infty(\mathfrak{h}^*)$ via $X(v) = \langle v, X \rangle$ where $v \in \mathcal{O}_\mu \subset \mathfrak{h}^*$. The Hamiltonian vector field for such an observable is $\pi_{\text{co}}(X')$, where π_{co} in this context means the derived Lie algebra representation of the coadjoint action of H . Clearly, such a vector is a Killing vector since the metric on \mathfrak{h}^* (induced from the one on \mathfrak{h}) is invariant under the coadjoint action of H .

Now the symplectic leaf $P_{\mathcal{O}_\mu} = N^\# \times_H \mathcal{O}_\mu$ is locally a product of [a subset of] the cotangent bundle T^*Q and the coadjoint orbit \mathcal{O}_μ . Similarly, the polarization is locally a product of a vertical polarization and a Kähler polarization. Thus, using (4.16) and the comments above on the observables that can be quantized for a Kähler polarization, we see that, crucially, the general form of a classical observable which can be quantized to give a self-adjoint operator agrees with that given in (2.37). This agreement between the prediction, made in Section 2.4 via the use of a momentum map, of which classical observables should be quantized and the actual observables which can be quantized via the geometric quantization technique is striking and indeed most reassuring.

The quantum operator corresponding to the observable given in (2.37) acts on sections of $B_K = B \otimes \delta_D$ and can be found using (4.15). However, there is a much more elegant way to present the quantum operators, namely as the Lie algebra representation derived from a representation of a Lie group. We now explain this approach.

4.4. Polarized sections of the prequantum line bundle

In order to make the connection with induced representations we must first represent the polarized sections of B in a more transparent manner.

To begin with, consider the line bundle $B' = N \times_{H_\mu} \mathbb{C} \rightarrow N/H_\mu$. Now $N/H_\mu \simeq N \times_H (H_\mu \setminus H) \simeq N \times_H \mathcal{O}_\mu$ and we can represent sections of B' by functions $\psi : N \times H \rightarrow \mathbb{C}$ satisfying

$$\begin{aligned} \psi(n, h') &= \psi(nh, h'h) \quad \forall h \in H; \\ \psi(n, h_\mu h') &= \chi_\mu(h_\mu)\psi(n, h') \quad \forall h_\mu \in H_\mu. \end{aligned} \tag{4.17}$$

From ψ we can define a function $\gamma : N \rightarrow \mathbb{C}$ via

$$\psi(n, h) = \psi(nh^{-1}, e) \equiv \gamma(nh^{-1}). \tag{4.18}$$

It is easy to see that $\gamma(nh_\mu) = \chi_\mu(h_\mu^{-1})\gamma(n)$. Hence γ satisfies (4.3) and thus ψ represents a section s of B' . We can pull s back to give a section of the prequantum line bundle B . The crucial point is how to realize the polarization condition on the ψ 's.

We can define a distribution on $N \times_H \mathcal{O}_\mu$ in a similar manner to that used in Section 4.2. Specifically we take the trivial (zero) distribution on N and the normal Kähler polarization on $H_\mu \setminus H \simeq \mathcal{O}_\mu$. The direct sum of the two distributions on $N \times \mathcal{O}_\mu$ projects to a distribution on $N \times_H \mathcal{O}_\mu$. The key point is that sections s of B' which satisfy the “pseudo-polarization” condition, $\nabla_{\bar{X}}s = 0 \forall X \in V_P(N \times_H \mathcal{O}_\mu)$, pullback to polarized sections of B . Further, all polarized sections occur in this way. Everything becomes clearer if we use local coordinates. Namely, if $\{q^a, p_a\}$ are local canonical coordinates for $T^*(N/H)$ and $\{z^i\}$ are local (complex) coordinates for \mathcal{O}_μ , then polarized sections of B are of the form $\phi(q, z)$, i.e., holomorphic in z . Clearly these correspond directly with sections of B' satisfying the pseudo-polarization condition.

In terms of the functions $\psi : N \times H \rightarrow \mathbb{C}$, if we set $\phi_n(h) = \psi(n, h)$ and regard $\phi_n : H \rightarrow \mathbb{C}$ then the condition that ψ will correspond to a polarized section of B is that ϕ_n represents a polarized section of $H \times_{H_\mu} \mathbb{C}$ (where $(h, z) \sim (h_\mu h, \chi_\mu(h_\mu)z)$, $h_\mu \in H_\mu$) with respect to the Kähler polarization on $H_\mu \setminus H$. Consequently, let \mathcal{H}_μ be the completion of the pre-Hilbert space of square-integrable polarized sections of $H \times_{H_\mu} \mathbb{C}$. Note that sections of $H \times_{H_\mu} \mathbb{C}$ are represented by functions $\phi : H \rightarrow \mathbb{C}$ satisfying

$$\phi(h_\mu h) = \chi_\mu(h_\mu)\phi(h) \quad \forall h_\mu \in H_\mu. \tag{4.19}$$

Thus we are in the same setting as that detailed in Section 3.2 and so we have an irreducible unitary representation π_μ of H on \mathcal{H}_μ .

We can then define $E = N \times_H \mathcal{H}_\mu$ (where $(n, v) \sim (nh, \pi_\mu(h^{-1})v)$) and sections of this bundle can be represented by functions $\psi : N \times H \rightarrow \mathbb{C}$ satisfying (4.17). Further, by construction, these functions correspond to polarized sections of B ; hence we have proved:

Theorem 4.3. *There is a one-to-one correspondence between the polarized sections of the prequantum bundle B and the sections of $E = N \times_H \mathcal{H}_\mu$.*

One advantage of identifying sections of the prequantum bundle B with sections of $N \times_H \mathcal{H}_\mu$ is that the latter bundle is closely related to induced representations as we shall see in the next section, but first we return to a matter alluded to at the end of Section 4.1.

4.4.1. *The generalized Aharonov–Bohm effect*

Wu and Yang [51] gave a description of a generalized (i.e., the gauge group is non-Abelian) Aharonov–Bohm effect in terms of a *nonintegrable phase factor*. This is the “generalized phase change” of the wave function of the particle on being parallel transported between two points with respect to the connection which represents the gauge field. The term “generalized phase change” is used because the nonintegrable phase factor acts via an irreducible representation of the gauge group on the wave function of the particle. This representation is, in general, not one-dimensional.

The Aharonov–Bohm effect for the gauge group $SU(2)$ has been studied [18] in terms of particles satisfying the Dirac equation, or its nonrelativistic limit, confirming the predictions of Wu and Yang. In the context of geometric quantization, the study of the generalized Aharonov–Bohm effect has been constrained to trying to classify the different “prequantizations” of the “isospin” bundle $N \rightarrow N/H_\mu$ [19]. This was found only to be possible when the relevant bundles were trivial. We will now show, by considering the bundle $E = N \times_H \mathcal{H}_\mu$ rather than the line bundle associated to $N \rightarrow N/H_\mu$, how the Wu and Yang nonintegrable phase factor appears naturally in our approach together with the representation of the gauge group H via which the phase factor acts on the wave function of the particle.

The bundle $E = N \times_H \mathcal{H}_\mu$ is an associated vector bundle of the principal bundle $N \rightarrow N/H$. Now this latter bundle has a connection α and thus there is a corresponding covariant derivative ∇^α on the sections of E . We will now show that this covariant derivative is equivalent to the one on the sections of the line bundle $B' = N \times_{H_\mu} \mathbb{C}$. This means it is permissible to consider parallel transport in E rather than in B' .

Let s be a section of E . We can represent s by $s(q) = [n(q), \psi(n(q), h)]_H$ where ψ satisfies (4.17) and $\pi_{N \rightarrow N/H}(n(q)) = q \in Q = N/H$. Now consider a curve $\sigma(t)$ in Q . We can choose n so that $n(\sigma(t)) = \tilde{\sigma}(t)$ is an arbitrary horizontal lift of $\sigma(t)$ with respect to the connection α . Let X be the tangent to $\sigma(t)$ at $t = 0$. Then

$$\nabla_X^\alpha s = \left[\tilde{\sigma}(0), \left. \frac{d}{dt} \psi(n(\sigma(t)), h) \right|_{t=0} \right]_H. \tag{4.20}$$

Now we saw earlier how sections of E could be identified with sections of B' . Here $s(q)$ corresponds to a section s' of B' where, with γ as defined in (4.18),

$$s'(q') = [n(q)h^{-1}, \gamma(n(q)h^{-1})]_{H_\mu}, \tag{4.21}$$

and $q' = \pi_{N \rightarrow N/H_\mu}(n(q)h^{-1})$. Let $\sigma'(t) = \pi_{N \rightarrow N/H_\mu}(\tilde{\sigma}(t)h^{-1})$, then

$$s'(\sigma'(t)) = [\tilde{\sigma}(t)h^{-1}, \gamma(n(\sigma(t))h^{-1})]_{H_\mu}. \tag{4.22}$$

It is easy to see that $\tilde{\sigma}(t)h^{-1}$ is a horizontal lift of $\sigma'(t)$ with respect to α' since if A is the tangent vector to $\tilde{\sigma}(t)$ (so $\alpha(A) = 0$), then $\alpha'(\rho_{h^{-1}*}A) = \text{pr}(\text{Ad}_h(\alpha(A))) = 0$. Thus, letting X' be the tangent vector to $\sigma'(t)$ at $t = 0$, we have

$$\nabla_{X'}^{\alpha'} s' = \left[\tilde{\sigma}(0)h^{-1}, \left. \frac{d}{dt} \gamma(n(\sigma(t))h^{-1}) \right|_{t=0} \right]_{H_\mu}; \tag{4.23}$$

and the right-hand side corresponds to the section $[\tilde{\sigma}(0), (d/dt)\psi(n(\sigma(t)), h)|_{t=0}]_H$ of E in agreement with (4.20).

Now let $\sigma(t)$ be a curve in Q with $\sigma(0) = q_0$ and $\sigma(1) = q_1$. Denoting, as before, $\tilde{\sigma}(t)$ to be the horizontal lift of $\sigma(t)$ to N with respect to the connection α , define $h \in H$ by

$$\tilde{\sigma}(1) = \tilde{\sigma}(0)h. \tag{4.24}$$

The general expression for h is

$$h = P \exp \left(- \int_{q_0}^{q_1} \mathcal{A}_a dq^a \right), \quad (4.25)$$

where P is a path-ordering operator along $\sigma(t)$ (which is necessary as H is, in general, non-Abelian) and \mathcal{A} is the local form on Q for α . This is the nonintegrable phase factor of Wu and Yang [51]. For a section $s = [n, v]_H$ of E , the change in $v \in \mathcal{H}_\mu$ on s being parallel transported around σ is given by $\pi_\mu(h)$, i.e.,

$$v \rightarrow \pi_\mu \left(P \exp \left(- \oint_\sigma \mathcal{A}_a dq^a \right) \right) v. \quad (4.26)$$

When $\sigma(t)$ is closed, i.e., $q_0 = q_1$, the “phase change” given in (4.26) is the generalized version of (4.8) and is the corresponding Aharonov–Bohm effect for arbitrary H . (Note that this result is in agreement with the case $H = U(1)$ considered in Section 4.1 since when H is Abelian $\pi_\mu = \chi_\mu$.)

4.5. Induced representations

The theory of induced representations is well known for the case where one starts with a representation π_μ of H acting on \mathcal{H}_μ and induces, from π_μ , a representation π^μ for a Lie group G where $H \subset G$. The induced representation π^μ acts on sections of the bundle $G \times_H \mathcal{H}_\mu$ (e.g., see [47,3]). Now a generalization of this type of induced representation, due to Moscovici [38], exists for the case in hand of the bundle N . The starting point is the bundle $E = N \times_H \mathcal{H}_\mu$, given in the previous section, with sections of E identified with functions $\Psi : N \rightarrow \mathcal{H}_\mu$ satisfying $\Psi(nh) = \pi_\mu(h^{-1})\Psi(n)$ for all $h \in H$. The representation π^μ of a group G' which acts on the left on N and commutes with the right action of H is given by

$$(\pi^\mu(g)\Psi)(n) = \Psi(g^{-1}n), \quad g \in G'. \quad (4.27)$$

We are naturally interested in taking $G' = \text{Aut } N$. (Note this is not a special case of [38] since it was assumed there that G' was locally compact.) We expect that, for $A \in \mathcal{L}(\text{Aut } N)$, the action given by $d\pi^\mu(A)$ corresponds to the observable $\hat{J}_\mu(A, 0)$ where $\hat{J}_\mu : \mathcal{L}(\text{Aut } N \times C^\infty(Q)) \rightarrow C^\infty(P_{\mathcal{O}_\mu})$ is defined just before (2.35) and $d\pi^\mu$ is defined, via (1.2), on the domain of compactly supported cross-sections of the vector bundle E . Before showing that this is the case, we remark that, as we will see, the group $C^\infty(Q)$ can be incorporated into π^μ in an obvious way to give a unitary representation of $\text{Aut } N \times C^\infty(Q)$. This representation is the same as that used by Landsman [30] except here the choice of such a representation is now fully justified in that we show that the derived Lie algebra representation corresponds to specific classical observables via the map \hat{J}_μ . Also we note that this representation of $\text{Aut } N \times C^\infty(Q)$ essentially appears in Isham [20, Ch. 5.2] under the guise of *lifted* group actions. Isham starts with a group action on Q and then

considers possible lifts of this to an automorphism of N . We have avoided the use of such lifted actions by starting with the group $\text{Aut } N$ to begin with.

The action of $\text{Aut } N$ on a function Ψ given via (4.27) corresponds to an action on the sections of the bundle $B' = N \times_{H_\mu} \mathbb{C}$. Specifically these sections of B' are represented by functions $\gamma : N \rightarrow \mathbb{C}$ satisfying $\gamma(nh_\mu) = \chi_\mu(h_\mu^{-1})\gamma(n)$ where $h_\mu \in H_\mu$. The action π^μ on these sections is then $(\pi^\mu(\phi)\gamma)(n) = \gamma(\phi^{-1}n)$. In terms of a section s of B' we have

$$(\pi^\mu(\phi)s)(q') = \phi s(\bar{\phi}_0^{-1}q'), \quad q' \in N/H_\mu. \tag{4.28}$$

Here, $\bar{\phi}_0$ denotes the diffeomorphism defined on N/H_μ in the same fashion as in (2.29). Returning to the convention for projection maps of bundles used in Section 2.3, sections s then pullback to give sections j^*s of the prequantum bundle $B = N^\# \times_{H_\mu} \mathbb{C}$, where $j = \pi_{N/H_\mu} \circ \iota$. Let π_0^μ denote the corresponding action of π^μ on these sections, i.e., $\pi_0^\mu(\phi)(j^*s) = j^*(\pi^\mu(\phi)s)$. Using the realization $N^\# = \{(n, p) : n \in N, p \in T_{\pi_{N \rightarrow Q}(n)}^* Q\}$ and denoting an element of $N^\#/H_\mu$ by $([n]_{H_\mu}, p_{\pi_{N \rightarrow Q}(n)})$, we find

$$\tau_{\phi^{-1}}^B((\pi_0^\mu(\phi)(j^*s))([n]_{H_\mu}, p_{\pi_{N \rightarrow Q}(n)})) = (j^*s)([\phi^{-1}n]_{H_\mu}, \bar{\phi}^* p_{\pi_{N \rightarrow Q}(n)}). \tag{4.29}$$

Here τ^B denotes the left action of ϕ on elements of $B = N^\# \times_{H_\mu} \mathbb{C}$ via $\tau_\phi^B[\beta_n, z]_{H_\mu} = [\tau_{(\phi,0)}\beta_n, z]_{H_\mu}$, with $\beta_n \in N^\# \subset T^*N$. The vector field V , generated by the infinitesimal action of $A \in \mathcal{L}(\text{Aut } N)$ on B via τ^B , intrinsically characterizes the classical observable to which the representation π_0^μ corresponds. We write $V = A^B$, where the superscript B denotes the space on which $\text{Aut } N$ is acting. Recall, that in Section 3.1, we gave the relation between an observable, its corresponding vector field on B and the resulting prequantum operator. (The term prequantum is used to emphasize that these operators are regarded as acting on general sections of B rather than the polarized ones.) We intend to use this relation to show:

Theorem 4.4. *The prequantum operator corresponding to the observable $\hat{J}_\mu(A, 0)$ is given by $\hbar d\pi_0^\mu(A)$.*

To begin with we return to (4.29) and note that we can write this as

$$\tau_\phi^B((\pi_0^\mu(\phi^{-1})(j^*s))([n]_{H_\mu}, p_{\pi_{N \rightarrow Q}(n)})) = (j^*s)(\tau_{(\phi,0)}([n]_{H_\mu}, p_{\pi_{N \rightarrow Q}(n)})). \tag{4.30}$$

This now corresponds to (3.4) since $\xi_{\hat{J}_\mu(A,0)} = A^{P_\mu}$. It now remains to show that the vector field A^B corresponds to the observable $\hat{J}_\mu(A, 0)$. The verification of this result is technical and we first present two lemmas.

Lemma 4.5.

$$(i) \quad \pi_{B \rightarrow P_\mu} A^B = \xi_{\hat{J}_\mu(A,0)}, \tag{4.31}$$

$$(ii) \quad \hbar \langle \tilde{\Theta}, A^B \rangle = \hbar \langle \tilde{\Theta}, A^B \rangle = -(\hat{J}_\mu(A, 0)) \circ \pi_{B \rightarrow P_\mu}. \tag{4.32}$$

Proof. Now $\pi_{B \rightarrow P_\mu} A^B$ is just the vector field generated by A acting on P_μ . Hence, from the properties of the momentum map \tilde{J}_μ , it is evident that (4.31) holds.

To verify (4.32), let $b : U \subset N/H_\mu \rightarrow N$ denote a section of the bundle $N \rightarrow N/H_\mu$. Then the section j^*b gives a local trivialization τ of B via $\tau(q', z) = [(s^*b)(q'), z]_{H_\mu}$. In this trivialization the connection one-form is given by

$$\tilde{\Theta} = \hbar^{-1} \pi_{B \rightarrow P_\mu}^* \Theta - i \tau^{-1*} \frac{dz}{z}, \tag{4.33}$$

where Θ is the local potential one-form of the connection given in (2.24). From the definition of θ_0 in (2.9), and recalling that elements (n, p) of $N^\#$ correspond to elements in the annihilator of the vertical bundle of TN , we have

$$\langle \pi_{B \rightarrow P_\mu}^* i^* \theta_0, A^B \rangle_{[(n,p),z]_{H_\mu}} = \langle \pi_{N \rightarrow QP}^* A^N \rangle_n. \tag{4.34}$$

The remaining part of Θ , as given in (2.24), consists of an α_μ term. Now

$$\langle \pi_{B \rightarrow P_\mu}^* j^* b^* \alpha_\mu, A^B \rangle_{[(n,p),z]_{H_\mu}} = \langle \pi_{N \rightarrow N/H_\mu}^* b^* \alpha_\mu, A^{B'} \rangle_n; \tag{4.35}$$

further,

$$\langle \tau^{-1*} \frac{dz}{z}, A^B \rangle_{[(n,p),z]_{H_\mu}} = \langle \tau_1^{-1*} \frac{dz}{z}, A^{B'} \rangle_{[n,z]_{H_\mu}}, \tag{4.36}$$

where τ_1 is the local trivialization of $N \times_{H_\mu} \mathbb{C}$ via $\tau_1(q, z) = [b(q), z]_{H_\mu}$. To complete the verification of (4.32) we use the following result.

Lemma 4.6.

$$\langle \tilde{\alpha}_\mu, A^{B'} \rangle_{[n,z]_{H_\mu}} = \langle \bar{\tilde{\alpha}}_\mu, A^{B'} \rangle_{[n,z]_{H_\mu}} = \hbar^{-1} \langle \alpha_\mu, A^N \rangle_n, \tag{4.37}$$

where $\tilde{\alpha}_\mu = \hbar^{-1} \pi_{N \rightarrow N/H_\mu}^* b^* \alpha_\mu - i \tau_1^{-1*} (dz/z)$ is the connection one-form on B' and $\bar{\tilde{\alpha}}_\mu$ is its complex conjugate.

Proof. This can be readily checked by considering, for example, the curve $b(q(t))e^{tA'}$ in N and the corresponding curve $[b(q(t))e^{tA'}, z]_{H_\mu}$ in B' ; here $A' \in \mathfrak{h}_\mu$. □

Thus, it finally remains to calculate $\langle \alpha_\mu, A^N \rangle_n$. We have

$$\langle \alpha_\mu, A^N \rangle_{s(q)h} = \langle \pi_{\text{co}}(\mathfrak{h}) \cdot \mu, \alpha_{s(q)}(A^N) \rangle, \tag{4.38}$$

where s is the section of the bundle $N \rightarrow N/H$ used at the end of Section 2.4. Combining this equation with (4.34) we finally obtain

$$\hbar \langle \tilde{\Theta}, V \rangle_{[(s(q)h),p],z]_{H_\mu}} = - \langle \pi_{N \rightarrow QP}^* A^N \rangle_{s(q)} - \langle \pi_{\text{co}}(\mathfrak{h}) \cdot \mu, \alpha_{s(q)}(A^N) \rangle. \tag{4.39}$$

Note that the right-hand side is in fact a function on $P_\mu \simeq P_{\mathcal{O}_\mu} = N^\# \times_H \mathcal{O}_\mu$. Now for $[\beta_{s(q)}, v]_H \in N_{s(q)}^\# \times_H \mathcal{O}_\mu$ we can write this as $[\rho_{h^{-1}}^* \beta_{s(q)}, \mu]_H \in N_{s(q)h}^\# \times_H \{\mu\}$ where h is such that $\pi_{\text{co}}(h) \cdot \mu = v$. So we can write

$$\begin{aligned} \hbar \langle \tilde{\Theta}, V \rangle_{[[\beta_{s(q)},v]_H],z]_{H_\mu}} &= - \langle \pi_{N \rightarrow QP}^* A^N \rangle_{s(q)} - \langle v, \alpha_{s(q)}(A^N) \rangle \\ &= - (\hat{J}_\mu(A, 0))[\beta_{s(q)}, v]_H \end{aligned} \tag{4.40}$$

using (2.36). Clearly the same expression is obtained for $\hbar\langle\tilde{\Theta}, V\rangle$ and thus this completes the proof of Lemma 4.5. \square

By considering (4.30), (3.5) and Lemma 4.5 we see that Theorem 4.4 has been proved.

It finally remains to incorporate $C^\infty(Q)$ into the representation π^μ defined in (4.28). For $(\phi, f) \in \text{Aut } N \times C^\infty(Q)$, following [30] we put

$$(\pi^\mu(\phi, f)s)(q') = \exp\{-i\hbar^{-1}f(\pi_{N/H_\mu \rightarrow Q}(q'))\}\phi s(\tilde{\phi}_0^{-1}q'), \tag{4.41}$$

which gives a [left] representation of $\text{Aut } N \times C^\infty(Q)$ on the sections of B' . Note that, in terms of the functions Ψ used in (4.27), the representation π^μ is given by

$$(\pi^\mu(\phi, f)\Psi)(n) = \exp\{-i\hbar^{-1}f \circ \pi_{N \rightarrow Q}(n)\}\Psi(\phi^{-1}n). \tag{4.42}$$

Concentrating on the case $\pi^\mu(0, f)$ we claim:

Theorem 4.7. *The prequantum operator corresponding to the observable $\hat{J}_\mu(0, f)$ is given by $\hbar d\pi_0^\mu(0, f)$.*

Identifying the Lie algebra of $C^\infty(Q)$ with the Lie group we have, in the notation of (3.4), $\rho_t[\beta_n]_{H_\mu} = [\beta_n - t\pi_{N \rightarrow Q}^*df]_{H_\mu}$. Now let $\hat{\rho}_t = \pi_0^\mu(0, -tdf)$, i.e., in a local trivialization of B

$$(\hat{\rho}_t j^*s)([n]_{H_\mu}, p) = ([n]_{H_\mu}, p, \exp\{i\hbar^{-1}tf \circ \pi_{N \rightarrow Q}(n)\}\psi([n]_{H_\mu})), \tag{4.43}$$

where s is a section determined locally by $\psi \in C^\infty(U \subset N/H_\mu)$, i.e., $(j^*s)([n]_{H_\mu}, p) = ([n]_{H_\mu}, p, \psi([n]_{H_\mu}))$. It follows that the corresponding δ_t is given by

$$\delta_t((j^*s)([n]_{H_\mu}, p)) = ([n]_{H_\mu}, p - tdf, \exp\{-i\hbar^{-1}tf \circ \pi_{N \rightarrow Q}(n)\}\psi([n]_{H_\mu})). \tag{4.44}$$

Regarding, as before, δ_t as the flow of the vector field $V = \xi^B(f)$ (i.e., the vector field on B generated by the action of f via τ) we now only need to prove:

Lemma 4.8.

$$(i) \quad \pi_{B \rightarrow P_\mu}^* \xi^B(f) = \xi_{j_\mu(0, f)}; \tag{4.45}$$

$$(ii) \quad \hbar\langle\tilde{\Theta}, \xi^B(f)\rangle = \hbar\langle\tilde{\Theta}, \xi^B(f)\rangle = -f \circ \pi_{B \rightarrow Q}. \tag{4.46}$$

Proof. As in Lemma 4.5, (4.45) follows from the properties of the momentum map \bar{J}_μ . Also it is easy to see from (2.9) that $\langle\pi_{B \rightarrow P_\mu}^*l^*\theta_0, \xi^B(f)\rangle = 0$. Thus it just remains to calculate $\langle\pi_{B \rightarrow P_\mu}^*j^*b^*\alpha_\mu, \xi^B(f)\rangle$. Using (4.33) we obtain

$$\hbar\langle\tilde{\Theta}, \xi^B(f)\rangle_{|(n, p), z]_{H_\mu}} = \hbar\langle\tilde{\Theta}, \xi^B(f)\rangle_{|(n, p), z]_{H_\mu}} = -f \circ \pi_{N \rightarrow Q}(n), \tag{4.47}$$

as required by (4.46). \square

In the same manner as Theorem 4.4 this completes the proof of Theorem 4.7. Note that $(\hbar d\pi_0^\mu(0, f)\psi)(q') = f(\pi_{P_\mu \rightarrow Q}(q'))\psi(q')$ as expected.

So far we have found operators which act on the sections of B . We must now restrict these to act on *polarized* sections of B . Recall that there is a one-to-one correspondence between the polarized sections of B and the sections of $E = N \times_H \mathcal{H}_\mu$. Thus, we see that the action of π_0^μ on the polarized sections is equivalent to the action of π^μ on the sections of E . So, to summarize, we can put our results for $\text{Aut } N$ and $C^\infty(Q)$ together to obtain our key result: the prequantum operator corresponding to the classical observable $\hat{J}_\mu(A, f)$ is given by $\hbar d\pi^\mu(A, f)$. However, this is not quite the complete picture because so far we have considered the quantum operators to be acting on sections of B rather than sections of $B_K = B \otimes \delta_D$. We address this point in the next section.

4.6. Unitary representations

As it stands, the representation π^μ fails to be unitary unless there exists a measure on N which is invariant under $\text{Aut } N$. This can be overcome in a standard way using the Radon–Nikodym derivative, e.g., [20, Ch. 5.2]. Let η be an H -invariant measure on N , which in turn determines a measure ν on Q . Then we define the representation π^μ by replacing (4.42) with

$$(\pi^\mu(\phi, f)\Psi)(n) = \left(\frac{d\eta(\phi^{-1}n)}{d\eta(n)}\right)^{1/2} \exp\{-i\hbar^{-1} f \circ \pi_{N \rightarrow Q}(n)\}\Psi(\phi^{-1}n). \tag{4.48}$$

This then gives a unitary representation of $\text{Aut } N \times C^\infty(Q)$. The addition of the square-root term corresponds to the replacement of B by B_K in Section 4.3 and the fixing of a choice of η such that $\eta\bar{\eta} = d\nu$. The inner product is (cf. 4.14)

$$(\Psi, \Psi') = \int_Q d\nu(\pi_{N \rightarrow Q}(n)) (\Psi(n), \Psi'(n))\mathcal{H}_\mu, \tag{4.49}$$

and we restrict our attention to smooth functions Ψ that have compact support.

Further the representation π^μ is, in general, irreducible. To see this we first recall that there is a one-to-one correspondence between the sections of E and the functions $\gamma : N \rightarrow \mathbb{C}$ which satisfy (4.3) and the “pseudo-polarization” condition detailed in Section 4.4. In terms of the γ ’s we have

$$(\pi^\mu(\phi, f)\gamma)(n) = \left(\frac{d\eta(\phi^{-1}n)}{d\eta(n)}\right)^{1/2} \exp\{-i\hbar^{-1} f \circ \pi_{N \rightarrow Q}(n)\}\gamma(\phi^{-1}n). \tag{4.50}$$

For a general Lie group G , the representation $\hat{\pi}$ of $G \times C^\infty(N)$ acting on smooth functions $\psi : N \rightarrow \mathbb{C}$, given by

$$(\hat{\pi}(g, f)\psi)(n) = \left(\frac{d\eta(g^{-1}n)}{d\eta(n)}\right)^{1/2} e^{-if(n)}\psi(g^{-1}n) \tag{4.51}$$

is irreducible [20] provided N does not decompose into a disjoint union of two G -invariant subsets both of which have positive η -measure. (This situation can be overcome if η is

required to be G -ergodic.) Returning to our representation π^μ we see that it is closely related to $\hat{\pi}$ except that firstly we are considering a subspace of $C^\infty(N, \mathbb{C})$, i.e., the functions γ satisfying the conditions noted above. This does not alter the irreducibility of $\hat{\pi}$. Secondly, the function f in (4.50) is lifted to one on N which means that it cannot vary along the fibres of $N \rightarrow N/H$. However, this restriction merely ensures that the action of π^μ is to create a function which satisfies the conditions on the γ 's. Hence π^μ is irreducible, provided N does not decompose into a disjoint union of two $\text{Aut } N$ invariant subsets.

Landsman [30] has given an explicit form of the representation π^μ in terms of functions $\psi^\alpha : U_\alpha \subset Q \rightarrow \mathcal{H}_\mu$. Specifically, cover Q with open sets $\{U_\alpha\}$ and denote local smooth sections of N by $s_\alpha : U_\alpha \rightarrow N$ such that for $q \in U_\alpha \cap U_\beta$, $s_\beta(q) = s_\alpha(q)g_{\alpha\beta}(q)$ where $g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow H$ is the transition function for the two coordinate patches U_α and U_β . An element of \mathcal{H}^μ is represented by a collection $\{\psi_\alpha\}$ of smooth functions $\psi_\alpha : U_\alpha \rightarrow \mathcal{H}_\mu$, which are related on $U_\alpha \cap U_\beta$ by

$$\psi_\alpha(q) = \pi_\mu(g_{\alpha\beta}(q))\psi_\beta(q). \tag{4.52}$$

The action of π^μ on these functions is given by

$$\begin{aligned} (\pi^\mu(\phi, f)\psi_\alpha)(q) &= \left(\frac{dv(\bar{\phi}^{-1}q)}{dv(q)}\right)^{1/2} \\ &\times e^{-i\hbar^{-1}f(q)}\pi_\mu((h_\beta[\bar{\phi}^{-1}(s_\alpha(q))])^{-1})\psi_\beta(\bar{\phi}^{-1}q), \end{aligned} \tag{4.53}$$

where h_β is the element of H satisfying $s_\beta(\bar{\phi}^{-1}q)h_\beta = \phi^{-1}(s_\alpha(q))$. Here it is assumed that $q \in U_\alpha$ and $\bar{\phi}^{-1}q \in U_\beta$. Landsman [30] also gives a formula for the derived representation $d\pi^\mu$, which in our notation is

$$\begin{aligned} \hbar(d\pi^\mu(A, f)\psi_\alpha)(q) &= f(q)\left(-i\hbar[\nabla_{\pi_N \rightarrow Q^*A} + \frac{1}{2}\text{div}(\pi_N \rightarrow Q^*A)(q)] \right. \\ &\left. + d\pi_\mu(\alpha_{s_\alpha(q)}(A))\right)\psi_\alpha(q). \end{aligned} \tag{4.54}$$

Here α is the connection on $N \rightarrow Q$ and ∇ is the corresponding covariant derivative via the representation π_μ of H on \mathcal{H}_μ . Also note that the div term agrees with that in (4.15). As noted by Landsman [30] the operator $d\pi^\mu$, restricted to $\mathcal{L}(\text{Aut } N \times C_c^\infty(Q))$ (where $C_c^\infty(Q)$ is the subspace of smooth functions on Q with compact support), is defined and essentially self-adjoint on the domain of compactly supported cross-sections of the bundle $E = N \times_H \mathcal{H}_\mu$. Further, the right-hand side of (4.54) is actually independent of the connection used. The motivation for writing (4.54) in this manner is that the third term on the right-hand side is the generalization of the Poincaré term in the angular momentum of a charged particle moving in the field of a magnetic monopole [29]; if A is a symmetry of the dynamics then this term is the contribution of the external gauge field to the conserved operator $d\pi^\mu(A, 0)$.

We have now proved our main result:

Theorem 4.9. *For the constrained mechanical system whose reduced phase space is P_μ , the quantum operator corresponding to the classical observable $\hat{J}_\mu(A, f)$ is given by*

$\hbar d\pi^\mu(A, f)$ and acts on compactly supported cross-sections of the bundle $E = N \times_H \mathcal{H}_\mu$. In terms of the quantizing map Q_\hbar , this is written as

$$Q_\hbar(\hat{J}_\mu(A, f)) = \hbar d\pi^\mu(A, f). \tag{4.55}$$

Note that as π^μ is, in general, irreducible we have satisfied the irreducibility requirements discussed in Section 1.

We now turn to the problem of finding the Hamiltonian for the quantum system. Unfortunately the classical Hamiltonian is not in the subclass of observables that we can quantize using Q_\hbar ; indeed this is a generic problem with geometric quantization when the Hamiltonian is not linear in momentum. However, Landsman [30] has shown that the quantum Hamiltonian H_\hbar is given by the gauge-covariant Laplacian on $E = N \times_H \mathcal{H}_\mu$. Specifically, H_\hbar determines the time-evolution of an operator $\hat{L} = Q_\hbar(\hat{J}_\mu(A, f))$ via

$$\hat{L}(t) = e^{it\hbar^{-1}H_\hbar} \hat{L} e^{-it\hbar^{-1}H_\hbar}, \tag{4.56}$$

and H_\hbar , which acts on sections of E , is given by

$$H_\hbar = -\frac{1}{2}\hbar^2 \nabla \cdot \nabla + V_0. \tag{4.57}$$

Here we have included the potential V_0 , which was defined at the end of Section 2.4, and we note that the gauge-covariant Laplacian is defined with respect to the connection α .

Locally, we can use $(h^1, \dots, h^{d_H}, q^{d_H+1}, \dots, q^{d_N})$ as coordinates on N , where $(q^{d_H+1}, \dots, q^{d_N})$ are coordinates on Q and (h^1, \dots, h^{d_H}) are coordinates on the fibre H . We can motivate H_\hbar as the Hamiltonian if the coordinates are chosen such that $\alpha(\partial/\partial q^\alpha) = 0$ and $dv = d^m q$. The latter condition means that $\det \mathbf{g} = 1$, where \mathbf{g} is the metric on Q . In the notation of (2.39), we then find that H_\hbar coincides with H'_\hbar , where

$$H'_\hbar = -\frac{1}{2}\hbar^2 Q_\hbar(\hat{J}_\mu(A_\alpha, 0)) \mathbf{g}^{\alpha\beta} Q_\hbar(\hat{J}_\mu(A_\beta, 0)) + \hat{J}_\mu(0, V_0). \tag{4.58}$$

Note that $\mathbb{I}^{IJ} Q_\hbar(\hat{J}_\mu(A_I, 0)) Q_\hbar(\hat{J}_\mu(A_J, 0)) = \mathbb{I}^{IJ} d\pi_\mu(T_I) d\pi_\mu(T_J)$ is a Casimir operator for H and, as the representation π_μ is irreducible, this is a constant which can be omitted from the Hamiltonian. Hence H'_\hbar can be considered to be the quantum operator corresponding to $H_{\mathcal{O}_\mu}$ given in (2.39). To see that H_\hbar agrees with H'_\hbar it is easier to use the representation π^μ as defined on functions Ψ used in (4.27). Then the action of $d\pi^\mu$ is given by [30]

$$\hbar(d\pi^\mu(A, 0)\Psi)(n) = -i\hbar \left(\left(A + \frac{1}{2} \operatorname{div} A \right) \Psi \right) (n). \tag{4.59}$$

Thus, noting that $\operatorname{div}(\partial/\partial q^\alpha) = 0$, we find

$$H'_\hbar = -\frac{1}{2}\hbar^2 \frac{\partial}{\partial q^\alpha} \mathbf{g}^{\alpha\beta}(q) \frac{\partial}{\partial q^\beta}, \tag{4.60}$$

whilst, up to a constant, H_\hbar acting on the functions Ψ is given by $-\frac{1}{2}\hbar^2 \Delta_{\text{LP}}$, where Δ_{LP} is the Laplace–Beltrami operator. Thus, in this choice of coordinate system, H_\hbar agrees with H'_\hbar .

4.7. Homogeneous spaces

When the bundle N is a finite-dimensional Lie group G (with $H \subset G$) the configuration space $Q = G/H$ is homogeneous. Isham [20] has considered quantization on such configuration spaces and in this section we relate our work to his. In particular we can explain two unresolved features of Isham's method. The first is the appearance of *inequivalent quantizations*, i.e., the discovery of many different quantum schemes resulting from the same classical system. The second is the presence of quantizations which appear to be unrelated to the original system. We find that the geometric quantization approach shows that the inequivalent quantizations of Isham's correspond to slightly different classical systems, and also that the seemingly unrelated quantizations of Isham's are indeed quantizations resulting from a completely different physical system. Note that the different quantizations Isham finds are unrelated to whether or not the configuration space is multiply connected. We conclude the section with a worked example for the case $G = \text{SU}(2)$ and $H = \text{U}(1)$. This gives the homogeneous configuration space S^2 .

Let V be a vector space which carries an almost faithful representation of G and for which there is a G -orbit in V that is diffeomorphic to G/H . Isham [20] argues that quantization corresponds to representations of the semi-direct product group $\mathcal{G} = G \ltimes V^*$. Crucially Isham considers \mathcal{G} as a subgroup of $\text{Diff } Q \times C^\infty(Q)/\mathbb{R}$ (where \mathbb{R} denotes the functions constant on Q) and the phase space of the system to be T^*Q . A momentum map for the action of \mathcal{G} on T^*Q is found and indeed corresponds to the restriction of the momentum map $\tilde{J}_{\mu=0}$ of Section 2.4 to $\mathcal{G} \subset \text{Aut } N \times C^\infty(Q)$. Isham quantizes the system by finding irreducible unitary representations of \mathcal{G} (via Mackey theory) and using the momentum map to match observables on T^*Q with the generators of the representations of \mathcal{G} .

We can split the irreducible unitary representations of \mathcal{G} into two classes, those which arise from consideration of a G -orbit $\Theta \subset V$ where $\Theta \simeq G/H$ (the first class) and those from a G -orbit $\Theta' \subset V$ where $\Theta' \simeq G/H'$ with $H \not\simeq H'$ (the second class). We can now compare Isham's results to our own. Specifically the representations π^μ we find are the same as those in Isham's first class. (Here we are restricting π^μ to \mathcal{G} .) Crucially, however, each of our representations corresponds, via μ , to a different symplectic leaf in $(T^*G)/H$. Further, each symplectic leaf has a different momentum map and thus each of the different representations corresponds to a slightly different physical system. In terms of the quantizing map Q_\hbar , Isham considers the phase space $G^\# / H \simeq T^*Q$ with

$$Q_\hbar(\hat{J}_{\mu=0}(A, u)) = \hbar d\pi^\mu(A, u), \quad (4.61)$$

where $(A, u) \in \mathcal{L}(\mathcal{G})^* \simeq \mathfrak{g} \times V^*$. Note that it is not clear which representation π^μ is to be chosen on the right-hand side. Whereas we have the phase space $G^\# \times_H \mathcal{O}_\mu$ with

$$Q_\hbar(\hat{J}_\mu(A, u)) = \hbar d\pi^\mu(A, u). \quad (4.62)$$

It is now clear that different representations of \mathcal{G} correspond to different physical systems. In fact, for a particle moving in a Yang–Mills field, the different representations of \mathcal{G} correspond to the different possible charges that the particle could have.

The representations of \mathcal{G} in Isham’s second class clearly correspond to the quantizations of constrained systems which have H' as the symmetry (gauge) group. In terms of a particle in a Yang–Mills field, these representations correspond to a particle on the configuration space G/H' where the internal charge couples to the gauge group H' . Thus, they are unrelated to the original system.

4.7.1. The canonical connection

There is a natural choice for the metric on G ; specifically, as H is compact, \mathfrak{h} is reductive in \mathfrak{g} and a positive definite inner product $\langle \langle \cdot, \cdot \rangle \rangle$ exists on \mathfrak{g} which is invariant by $\pi_{\text{ad}}(H)$ (e.g., [24]). Thus, by defining \mathfrak{m} to be the orthogonal complement to \mathfrak{h} , with respect to this inner product, we have that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$, i.e., the decomposition is reductive. We can use this inner product on \mathfrak{g} to define one on $T_g G$ via $\langle \langle X, Y \rangle \rangle_g = \langle \langle \lambda_{g^{-1}*} X, \lambda_{g^{-1}*} Y \rangle \rangle$, thus defining a metric on G . We saw in Section 2.2 that a choice of a metric on N was equivalent to choosing a connection on $N \rightarrow N/H$. In this section we will explicitly identify this connection.

Let $\mathfrak{g}_{ab} = \langle \langle T_a, T_b \rangle \rangle$ where $\{T_i\}$ is a basis for \mathfrak{g} . We can write the Hamiltonian as

$$H_0(g, p) = \frac{1}{2} \mathfrak{g}^{ab} p_a p_b + V(g), \tag{4.63}$$

where $\mathfrak{g}^{ab} \mathfrak{g}_{bc} = \delta_c^a$ and the $\{p_i\}$ are coordinates on $T_g^* G$ in the left trivialization (2.12). The corresponding Legendre transformation $\mathbb{F}L$ is $(g, v^j) \rightarrow (g, p_j)$ where $p_j = \mathfrak{g}_{ja} v^a$ and $(g, v) \in G \times \mathfrak{g}$ represents $\lambda_{g*} v \in T_g G$.

We denote the momentum map for the right action of H on T^*G by J_H . From (2.10) we find for, $X \in \mathfrak{h}$,

$$\langle J_H(g, p), X \rangle = \langle p, X \rangle. \tag{4.64}$$

So $J_H^I(g, p) = p^I$ where $I = 1, \dots, d_H$. From the definition of \mathbb{I} (2.16) we have $\mathbb{I}(g)_{IJ} = \mathfrak{g}_{IJ}$. To calculate $\alpha^I(g, v)$, note that the choice of a reductive decomposition means that $g^{I\beta} = 0$, for $\beta = d_H + 1, \dots, d_G$. Thus, we readily find

$$\alpha^I(g, v) = v^I. \tag{4.65}$$

We see that α is the canonical connection on $G \rightarrow G/H$. The canonical connection ω for this bundle is defined to be the \mathfrak{h} component of the canonical (Maurer–Carter) one-form on G with respect to the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ (e.g., see [23]). Explicitly, $\omega = T_l \otimes \theta^I$, where $\{\theta^I\}$ are left invariant one-forms as defined in Section 2.1 using a basis of \mathfrak{g}^* which is dual to the basis $\{T_i\}$ of \mathfrak{g} , i.e., $\langle d^a, T_b \rangle = \delta_b^a$. If $(g, v)_L \in G \times \mathfrak{g}$ represents the point $\lambda_{g*} v \in T_g G$ then we have $\omega(\lambda_{g*} v) = T_l v^I$, i.e., $\omega^I = v^I$, in agreement with (4.65). The one-form α_μ on G is then just

$$\alpha_\mu = \mu_I \theta^I. \tag{4.66}$$

It is easy to see that the trivialization of $T^*G \simeq G^\# \times \mathfrak{h}^*$ induced by (2.26) corresponds to the left trivialization of T^*G in (2.12) where $G^\# \simeq G \times \mathfrak{m}^*$. This allows us to rewrite (2.32) as

$$\langle J(g, p), (X, u) \rangle = \langle \pi_{\text{co}}(\mathfrak{g}) \cdot p, X \rangle + \langle u, \sigma(g)a \rangle, \tag{4.67}$$

where $(X, u) \in \mathcal{L}(\mathcal{G}) \simeq \mathfrak{g} \times V^*$ and σ is the representation of G on V . We regard J as a momentum map for the left action of \mathcal{G} on T^*G . In passing we note that as \mathcal{G} is finite dimensional, $\mathcal{L}(\mathcal{G})^*$ is foliated by symplectic leaves which are the coadjoint orbits. If the explicit form for the coadjoint action of \mathcal{G} on $\mathcal{L}(\mathcal{G})^*$ is considered (e.g., see [34]) it is easy to see that the coadjoint orbit $\pi_{\text{co}}(\mathcal{G}) \cdot (v, a)$, where $(v, a) \in \mathfrak{g}^* \times V$, is contained in $(\mathfrak{g}^*, \sigma(G)a)$ which is exactly $J(T^*G)$. Hence the symplectic leaves in $J(T^*G)$ are coadjoint orbits. Thus, by the arguments of Section 2.4, \bar{J}_μ is a symplectic diffeomorphism which maps the symplectic leaf $G \times_H (\mathfrak{m}^* \times \mathcal{O}_\mu)$ to the symplectic leaf $\pi_{\text{co}}(\mathcal{G}) \cdot (v, a)$ where $v \in \mathfrak{g}^*$ such that $v|_{\mathfrak{h}} = \mu \in \mathfrak{h}^*$. Explicitly,

$$\bar{J}_\mu[g, p]_H = (\pi_{\text{co}}(\mathfrak{g}) \cdot p, \sigma(g)a) \in \mathcal{L}(\mathcal{G})^* \simeq \mathfrak{g}^* \times V, \tag{4.68}$$

where $[g, p]_H \in G \times_H (\mathfrak{m}^* \times \mathcal{O}_\mu)$. This gives an elegant alternative proof of a previously known result [34].

4.7.2. The case $G = \text{SU}(2)$, $H = \text{U}(1)$

We can parametrize $\text{SU}(2)$ using the Euler angles (ϕ, θ, χ) :

$$g(\phi, \theta, \chi) = e^{-\phi a_3} e^{-\theta a_2} e^{-\chi a_3}, \tag{4.69}$$

where $a_j = \frac{1}{2}i\sigma_j$, the $\{\sigma_j\}$ are the Pauli spin matrices, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$ and $0 \leq \chi \leq 4\pi$. We regard $H = \text{U}(1)$ as the subgroup

$$H = \{g(0, 0, \chi) : 0 \leq \chi \leq 4\pi\}. \tag{4.70}$$

There is a standard homomorphism (e.g., see [8]) $\alpha : \text{SU}(2) \rightarrow \text{SO}(3)$ given by $\alpha(u)_{jk} = \frac{1}{2}\text{tr}(\sigma_j u \sigma_k u^{-1})$. In the parametrization above this gives

$$G(\phi, \theta, \chi) = \alpha(g(\phi, \theta, \chi)) = e^{-\phi A_3} e^{-\theta A_2} e^{-\chi A_3}. \tag{4.71}$$

Here $[A_i]_{jk} = -\epsilon_{ijk}$ and (4.71) is the standard Euler angle parametrization of $\text{SO}(3)$. The subgroup $\text{U}(1) \subset \text{SU}(2)$ is mapped by α to the subgroup $G(0, 0, \chi) \simeq \text{SO}(2)$ whilst the kernel of α is just $\pm 1 \subset \text{U}(1)$. Hence we can see that α drops to a map on the quotient spaces $\text{SU}(2)/\text{U}(1) \rightarrow \text{SO}(3)/\text{SO}(2)$. This is readily observed to be a diffeomorphism; thus $\text{SU}(2)/\text{U}(1) \simeq S^2$. We can use α to give a representation of $\text{SU}(2)$ on \mathbb{R}^3 . Clearly the orbits of $\text{SU}(2)$ in \mathbb{R}^3 are then spheres (or just the origin). Thus by choosing $G = \text{SU}(2)$, $H = \text{U}(1)$ and $V = \mathbb{R}^3$ so that $\mathcal{G} = \text{SU}(2) \ltimes \mathbb{R}^{3*}$ we satisfy Isham's requirements for \mathcal{G} . We use the measure $dv = \sin \theta d\phi \wedge d\theta$ on S^2 . Note that this is G -invariant.

It is a standard result that the derived map $\alpha' : \mathfrak{su}(2) \rightarrow \mathfrak{so}(3)$ is an isomorphism with $\alpha'(a_i) = A_i$, e.g., see [9]. Further, the adjoint action π_{ad} of $\text{SO}(3)$ on $\mathfrak{so}(3) \simeq \mathbb{R}^3$ is the usual one, i.e., using $\{A_i : i = 1, 2, 3\}$ as a basis for $\mathfrak{so}(3)$ then if $p \in \mathbb{R}^3 \simeq \mathfrak{so}(3)$ and $g \in \text{SO}(3)$ then $\pi_{\text{ad}}(g) \cdot p = g \cdot p$, e.g., see [1]. Similarly regarding $\mathfrak{so}(3)^* \simeq \mathbb{R}^3$, we have $\pi_{\text{co}}(\mathfrak{g}) \cdot p = g \cdot p$. We can now write down an explicit expression for (4.68). We have

$$\bar{J}_\mu[g, p]_H = (\alpha(g) \cdot p, \alpha(g) \cdot a_0), \tag{4.72}$$

where we have used the fact that $\alpha'(\pi_{\text{co}}(\mathfrak{g}) \cdot A) = \pi_{\text{co}}(\alpha(\mathfrak{g})) \cdot \alpha'(A)$. Here $a_0 = (0, 0, 1)^T$ so, as is standard, we are taking S^2 to have unit radius.

We can use (4.54) to find explicit expressions for the derived representation $d\pi^\mu$ where we are considering π^μ as representation for $\mathcal{G} \subset \text{Aut } N \ltimes C^\infty(Q)$. Of course, however, we must first find the representation π_μ of H which corresponds to the coadjoint orbit $\mathcal{O}_\mu \subset \mathfrak{h}^*$. Since H is Abelian the coadjoint orbits are just single points; $\mathcal{O}_\mu = \{\mu\}$. Finding the representation of H which corresponds to the orbit \mathcal{O}_μ is a trivial example of the standard problem discussed in Section 3.2, since H_μ , the isotropy group of H , is just H . Writing X_μ for the homomorphism χ_μ of Section 3.2, then, if the elements of H_μ are written as $e^{-\chi a_3}$, $0 \leq \chi \leq 4\pi$, we have from (4.5)

$$X_\mu(e^{-\chi a_3}) = e^{i\hbar^{-1}\mu\chi}, \quad 2n \in \mathbb{Z}, \tag{4.73}$$

where we are regarding $\mathfrak{h}^* \simeq \mathbb{R}$. Requiring X_μ to be a single-valued function on H_μ , we see that the integrality condition is

$$\mu = n\hbar, \quad 2n \in \mathbb{Z}. \tag{4.74}$$

The sections of the prequantum line bundle $B \rightarrow \mathcal{O}_\mu$ are identified with functions $\psi : H \rightarrow \mathbb{C}$ such that

$$\psi(hh') = X(h)\psi(h'), \quad h, h' \in H, \tag{4.75}$$

and we obtain a representation π_μ of H on the sections of B by pulling back the ψ 's under right translation. In this trivial case we can, as $H_\mu = H$, identify each ψ with some $z \in \mathbb{C}$ via $\psi(e) = z$ so $(\pi_\mu(h)\psi)(h') = \psi(h'h) = X(h)\psi(h')$. Hence

$$\pi_\mu(e^{-\chi a_3}) = e^{i\hbar\chi}. \tag{4.76}$$

Thus, as expected, π_μ is an irreducible unitary representation of H . In this simple case the process of finding a polarization does not arise since \mathcal{O}_μ is just a single point.

We now turn to finding the induced representation π^μ . We can cover S^2 with the standard coordinate patches

$$M_N = \{(\phi, \theta) \in S^2 : 0 \leq \theta < \frac{1}{2}\pi + \epsilon, 0 \leq \phi \leq 2\pi\}; \tag{4.77}$$

$$M_S = \{(\phi, \theta) \in S^2 : \frac{1}{2}\pi - \epsilon < \theta \leq 2\pi, 0 \leq \phi \leq 2\pi\}, \tag{4.78}$$

where $\frac{1}{2}\pi > \epsilon > 0$ and (ϕ, θ) are the standard spherical polar angles. Following Landsman [28], we choose continuous sections $s_{N/S} : M_{N/S} \rightarrow \text{SU}(2)$, namely

$$s_N(\phi, \theta) = g(\phi, \theta, -\phi); \tag{4.79}$$

$$s_S(\phi, \theta) = g(\phi, \theta, \phi), \tag{4.80}$$

so that s_N is continuous at the North pole ($\theta = 0$) while s_S is at the South pole ($\theta = \pi$). On the overlap region $M_N \cap M_S$

$$s_S(\phi, \theta) = s_N(\phi, \theta)e^{-2\phi a_3}. \tag{4.81}$$

Before calculating the explicit form for the induced representations we first find an expression for the classical observables in local coordinates on each coordinate patch. Note that the symplectic leaf $P_{\mathcal{O}_\mu} = G \times_H (\mathfrak{m}^* \oplus \{\mu\}) \simeq G \times_H \mathfrak{m}^*$.

A section s of $G/H \rightarrow G$ determines a trivialization of $G \times_H \mathfrak{m}^* \simeq T^*(G/H)$ via

$$[s(q)h, p]_H = [s(q), h \cdot p]_H \tag{4.82}$$

$$\rightarrow (q, h \cdot p) \in G/H \times \mathfrak{m}^* \tag{4.83}$$

as every $g \in G$ has a unique factorization $g = s(q)h$. So $h \cdot p$ represents a one-form $p_1\theta^1 + p_2\theta^2 = p_\phi d\phi + p_\theta d\theta$ at $q = q(\phi, \theta)$. Thus we need to find $\{\theta^i_{s_{N/S}(\phi, \theta)}\}$. Starting with $s_N(\phi, \theta)$, we find that the local form for the left-invariant one-forms is

$$\begin{pmatrix} \theta^1_{s_N(\phi, \theta)} \\ \theta^2_{s_N(\phi, \theta)} \\ \theta^3_{s_N(\phi, \theta)} \end{pmatrix} = H'(\phi) \begin{pmatrix} -\sin \theta d\phi \\ d\theta \\ \cos \theta d\phi + d\chi \end{pmatrix}, \tag{4.84}$$

where $H'(\phi) = G(0, 0, \phi)$. Identifying $p = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$ with $\begin{pmatrix} p_1 \\ p_2 \\ p_3=0 \end{pmatrix} \in \mathbb{R}^3$ we have,

$$H'(-\phi) \cdot p = p', \quad \text{where } p' = \begin{pmatrix} -p_\phi / \sin \theta \\ p_\theta \end{pmatrix}.$$

Given $(p_\theta(q), p_\phi(q))$ we find the corresponding element of $G \times_H \mathfrak{m}^*$, using either s_N or s_S , is $[g(\phi, \theta, 0), p']_H$. We can now give (4.72) explicitly; setting $p'_3 = \mu$ so we are identifying $G \times_H \mathfrak{m}^*$ with $G \times_H (\mathfrak{m}^* \times \{\mu\})$, we have

$$\begin{aligned} \bar{J}_\mu(p_\theta(q), p_\phi(q)) &= (G(\phi, \theta, 0) \cdot p', G(\phi, \theta, 0) \cdot a_0) \\ &= \left(\begin{pmatrix} -p_\phi \cos \phi \cot \theta - p_\theta \sin \phi + \mu \sin \theta \cos \phi \\ -p_\phi \sin \phi \cot \theta + p_\theta \cos \phi + \mu \sin \theta \sin \phi \\ p_\phi + \mu \cos \theta \end{pmatrix}, \begin{pmatrix} \sin \theta \cos \phi \\ \sin \theta \sin \phi \\ \cos \theta \end{pmatrix} \right). \end{aligned} \tag{4.85}$$

Returning to the actual representations of \mathcal{G} themselves, we really require the derived representations. Let $\{u^i\}$ be the canonical basis for \mathbb{R}^3 . Using (4.54) and setting $\hat{q}^i = \hbar d\pi^\mu(0, u^i)$, immediately we find for $\psi^{N/S} \in L^2(M_{N/S}, \mathbb{C})$,

$$\hat{q}^i \psi^{N/S}(\phi, \theta) = q^i \psi^{N/S}(\phi, \theta), \tag{4.86}$$

where $q(\phi, \theta) = G(\phi, \theta, 0) \cdot a_0$. Finding the expression $d\pi^\mu(X, 0)$ requires some calculation. However, Landsman [28] has already done this for the very similar case of $\mathcal{G} = \text{SO}(3) \ltimes \mathbb{R}^3$ so we will not give the details. Recalling that $\mu = n\hbar$, we find for $\hat{L}_i^n = d\pi^\mu(a_i, 0)$

$$\begin{aligned}
 (\hat{L}_1^n \psi^{N/S})(\phi, \theta) = & \left[\left(i \cos \phi \cot \theta \frac{\partial}{\partial \phi} + i \sin \phi \frac{\partial}{\partial \theta} - n \frac{\cos \phi}{\sin \theta} (1 \mp \cos \theta) \right) \right. \\
 & \left. \times \psi^{N/S} \right](\phi, \theta); \tag{4.87}
 \end{aligned}$$

$$\begin{aligned}
 (\hat{L}_2^n \psi^{N/S})(\phi, \theta) = & \left[\left(i \sin \phi \cot \theta \frac{\partial}{\partial \phi} - i \cos \phi \frac{\partial}{\partial \theta} - n \frac{\sin \phi}{\sin \theta} (1 \mp \cos \theta) \right) \right. \\
 & \left. \times \psi^{N/S} \right](\phi, \theta); \tag{4.88}
 \end{aligned}$$

$$(\hat{L}_3^n \psi^{N/S})(\phi, \theta) = \left[\left(-i \frac{\partial}{\partial \phi} \mp n \right) \psi^{N/S} \right](\phi, \theta). \tag{4.89}$$

The term div in (4.54) vanishes because if the vector field X is complete, $\text{div } X = 0$ by the G -invariance of ν (e.g., see [1]). Also note that in the region the coordinate systems overlap, ψ^N and ψ^S are related by (4.52), namely

$$\psi^S(\phi, \theta) = e^{2in\phi} \psi^N(\phi, \theta). \tag{4.90}$$

The action of the generators of $SU(2)$, detailed above, agree with those given by Landsman [28] for $SO(3)$ except that here half-integer values of n are allowed, which reflects the fact that we are using $SU(2)$ rather than $SO(3)$.

We can now give the quantization explicitly. Using (4.85) together with (4.55), we have

$$\begin{aligned}
 -p_\phi \cos \phi \cot \theta - p_\theta \sin \phi + \mu \sin \theta \cos \phi & \rightarrow \hbar \hat{L}_1^n, \\
 -p_\phi \sin \phi \cot \theta + p_\theta \cos \phi + \mu \sin \theta \sin \phi & \rightarrow \hbar \hat{L}_2^n, \\
 p_\phi + \mu \cos \theta & \rightarrow \hbar \hat{L}_3^n, \\
 \sin \theta \cos \phi & \rightarrow \hat{q}^1, \\
 \sin \theta \sin \phi & \rightarrow \hat{q}^2, \\
 \cos \theta & \rightarrow \hat{q}^3,
 \end{aligned} \tag{4.91}$$

where the action of $\{\hat{q}^i\}$ and $\{\hat{L}_i\}$ on the respective coordinate patches is given in (4.86) and (4.87)–(4.89). Note that this is the quantization obtained when π^μ is restricted to \mathcal{G} and thus (4.91) does not give the quantum operators for all the observables that could be quantized. It is, of course, straightforward to calculate from (4.54) the quantum operators corresponding to these other observables but, for simplicity, we have just restricted ourselves to the ones corresponding to \mathcal{G} via \hat{J}_μ .

5. Conclusion

We have now achieved our aim of matching a preferred class of observables with their quantum operator counterparts in a way which satisfies the quantum conditions (Qi)–(Qiii) given in Section 1, i.e., Eq. (4.55). Although having used the method of geometric quantization we have managed to cast our results for the quantum operators in the language of representations rather than the form, given in (3.2), which is usually generated by the geometric quantization approach. Thus, our results can be considered to be a generalization of Isham's [20] approach (*modified* in view of the results of Section 4.7) in that, firstly, they are applicable in the case of a nonhomogeneous configuration space; and secondly, we now have a representation of $\text{Aut } N \times C^\infty(Q)$ rather than $G \times V^* \subset \text{Aut } N \times C^\infty(Q)$ and we thus have a correspondingly larger class of physical observables that can be quantized.

Finally, for a particle in an external Yang–Mills field, the role of the connection α is now transparent. Note that we can regard α to be given by the classical Hamiltonian H_0 on T^*N since H_0 implicitly gives the metric on N which then determines α . Firstly, the obvious role of the connection is in the quantum Hamiltonian H_h . Turning to the quantizing map Q_h , however, we see that this map is *independent* of the connection. This follows since, for the symplectic leaves of $(T^*N)/H$, we could, given $\mathcal{O}_\mu \in \mathfrak{h}^*$, write the corresponding symplectic leaf as $(J^{-1}(\mathcal{O}_\mu))/H$ which is defined without recourse to the connection. This is the reduced phase space of the particle. Similarly both the map \hat{J}_μ and the derived representation $d\pi^\mu$ are defined without regard to the connection. (Recall that right-hand side of (4.54) is independent of the connection used.) Thus, as claimed, the quantizing map is independent of the connection and, in fact, there is only one set of quantum operators, labelled by $\mathcal{L}(\text{Aut } N \times C^\infty(Q))$. Where the connection comes in, is that it allows the “external” and “internal” classical variables of the particle to be explicitly identified, i.e., it determines the local form of \hat{J}_μ given in (2.37) where the (q^α, p_β) are considered as “external” variables and v represents the “internal” variables. Thus, the connection determines the way in which the quantum operators are interpreted at a physical level.

Acknowledgements

The author would like to greatly thank N.P. Landsman for not only suggesting this line of research but also for his many helpful comments and suggestions. Additionally, N. Linden is thanked for several useful discussions.

References

- [1] R. Abraham and J.E. Marsden, *Foundations of Mechanics*, 2nd Ed. (Addison-Wesley Redwood City, 1985).
- [2] Y. Aharonov and D. Bohm, Significance of electromagnetic potentials, *Phys. Rev.* 115 (1959) 485–491.
- [3] A.O. Barut and R. Raczka, *Theory of Group Representations and Applications*, 2nd Ed. (revised) (World Scientific, Singapore, 1986).

- [4] R.J. Baston and M.G. Eastwood, *The Penrose Transform: Its Interaction with Representation Theory* (Oxford University Press, Oxford, 1989).
- [5] R.J. Blattner, Quantization and representation theory, *Proc. Symp. Pure Math.* 26 (1973) 147–165.
- [6] M. Blau, On the geometric quantisation of constrained systems, *Classical Quantum Gravity* 5 (1988) 1033–1044.
- [7] Y. Choquet-Bruhat and C. DeWitt-Morette, *Analysis, Manifolds and Physics. Part II* (North-Holland, Amsterdam, 1989).
- [8] J.F. Cornwell, *Group Theory in Physics*, Vol. 1 (Academic Press, London, 1984).
- [9] J.F. Cornwell, *Group Theory in Physics*, Vol. 2 (Academic Press, London, 1984).
- [10] C. Duval, J. Elhadad, M.J. Gotay, J. Sniatycki and G.M. Tuynman, Quantization and bosonic BRST theory, *Ann. Phys.* 206 (1991) 1–26.
- [11] C. Duval, J. Elhadad, M.J. Gotay and G.M. Tuynman, Nonunimodularity and the quantization of the pseudo-rigid body, in: *Proc. CRM Workshop on Hamiltonian Systems, Transformation Groups and Spectral Transform Methods*, Montréal, October 1989 (Les Publications du CRM).
- [12] M.J. Gotay, Constraints, reduction, and quantization, *J. Math. Phys.* 27 (1986) 2051–2066.
- [13] V. Guillemin and S. Sternberg, Multiplicity-free spaces, *J. Differential Geom.* 19 (1984) 31–56.
- [14] V. Guillemin and S. Sternberg, *Symplectic Techniques in Physics* (Cambridge University Press, Cambridge, 1984).
- [15] J. Harnad and J.E. Marsden, eds., *Hamiltonian Systems, Transformation Groups, and Spectral Transform Methods* (CRM Press, Montreal 1991).
- [16] P.A. Horváthy, Classical action, the Wu–Yang phase factor and prequantization, in: *Proc. Coll. Diff. Geom. Math. in Math. Phys.*, ed. J.-M. Souriau, *Lecture Notes in Math.*, Vol. 836 (Springer, Berlin, 1980).
- [17] P.A. Horváthy, Quantization in multiply connected spaces, *Phys. Lett.* 76A (1980) 11–14.
- [18] P.A. Horváthy, Non-Abelian Aharonov–Bohm effect, *Phys. Rev. D* 33 (1986) 407–414.
- [19] P.A. Horváthy and J. Kollár, The non-Abelian Aharonov–Bohm effect in geometric quantization, *Classical Quantum Gravity* 1 (1984) L61–L66.
- [20] C.J. Isham, Topological and global aspects of quantum theory, in: *Relativity, Groups and Topology 2*, eds. B.S. Dewitt and R. Stora, Les Houches Summer School, 1983 (North-Holland, Amsterdam, 1984).
- [21] D. Kazhdan, B. Kostant and S. Sternberg, Hamiltonian group actions and dynamical systems of Calogero type, *Comm. Pure Appl. Math.* 31 (1978) 481–508.
- [22] R. Kerner, Generalization of Kaluza–Klein theory for an arbitrary non-Abelian gauge group, *Ann. Inst. Henri Poincaré* 9 (1968) 143–152.
- [23] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Vol. I (Interscience, New York, 1963).
- [24] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Vol. II (Wiley/Interscience, New York, 1969).
- [25] B. Kostant, Orbits, symplectic structures, and representation theory, in: *Proc. US–Japan Seminar in Differential Geometry*, Kyoto, 1966 (Nippon Hyoronsha, Tokyo).
- [26] M. Kummer, On the construction of the reduced phase space of a Hamiltonian system with symmetry, *Indiana Univ. Math.* 30 (1981) 281–291.
- [27] N.P. Landsman, Rieffel induction as generalized quantum Marsden–Weinstein reduction, *J. Geom. Phys.* 15 (1995) 285–319; Erratum 17 (1995) 298.
- [28] N.P. Landsman, Quantization and superselection sectors. 2. Dirac monopole and Aharonov–Bohm effect, *Rev. Math. Phys.* 2 (1990) 73–104.
- [29] N.P. Landsman, Induced representations, gauge fields, and quantization on homogeneous spaces, *Rev. Math. Phys.* 4 (1992) 503–527.
- [30] N.P. Landsman, Strict deformation quantization of a particle in external gravitational and Yang–Mills fields, *J. Geom. Phys.* 12 (1993) 93–132.
- [31] G. Mackey, *Induced Representations* (Benjamin, New York, 1968).
- [32] C.M. Marle, Symplectic manifolds, dynamical groups and Hamiltonian mechanics, in: *Differential Geometry and Relativity*, eds. M. Cahen and M. Flato (Reidel, Dordrecht, Boston, 1976).
- [33] J.E. Marsden, *Lectures on Mechanics* (Cambridge University Press, Cambridge, 1992).
- [34] J.E. Marsden, T.S. Ratiu and A. Weinstein, Semi-direct products and reduction in mechanics, *Trans. Amer. Math. Soc.* 281 (1984) 147–177.

- [35] J.E. Marsden and J.E. Weinstein, Reduction of symplectic manifolds with symmetries, *Rep. Math. Phys.* 5 (1974) 121–130.
- [36] R. Montgomery, Canonical formulations of a classical particle in a Yang–Mills field and Wong’s equations, *Lett. Math. Phys.* 8 (1984) 59–67.
- [37] R. Montgomery, J.E. Marsden and T.S. Ratiu, Gauged Lie–Poisson structures, *Cont. Math. AMS* 28 (1994) 101–114.
- [38] H. Moscovici, Generalised induced representations, *Rev. Roum. Math. Pures et Appl.* 14 (1969) 1539–1551.
- [39] M. Nakahara, *Geometry, Topology and Physics* (Hilger, Bristol, 1990).
- [40] M. Puta, *Hamiltonian Mechanical Systems and Geometric Quantization* (Kluwer Academic Publishers, Dordrecht, 1993).
- [41] J.H. Rawnsley, Representations of a semi-direct product by geometric quantization, *Math. Proc. Cambridge Philos. Soc.* 78 (1975) 345–350.
- [42] M.A. Robson, *Geometric quantization of constrained systems*, Ph.D. Thesis, University of Cambridge, (1995).
- [43] D.J. Simms and N.M.J. Woodhouse, *Lectures on Geometric Quantization* (Springer, Berlin, 1976).
- [44] S. Smale Topology and mechanics, *Inv. Math.* 10, 11 (1970) 305–331, 45–64.
- [45] J. Sniatycki, *Geometric Quantization and Quantum Mechanics, Applied Mathematical Science Series, Vol. 30* (Springer, Berlin, 1980).
- [46] S. Sternberg, Minimal coupling and the symplectic mechanics of a classical particle in the presence of a Yang–Mills field, *Proc. Nat. Acad. Sci.* 74 (1977) 5253–5254.
- [47] V.S. Varadarajan, *Geometry of Quantum Theory*, 2nd Ed. (Springer, New York, 1985).
- [48] A. Weinstein, A universal phase space for particles in Yang–Mills fields, *Lett. Math. Phys.* 2 (1978) 417–420.
- [49] A. Weinstein, The local structures of Poisson manifolds, *J. Differential Geom.* 18 (1983) 523–557.
- [50] N.M.J. Woodhouse, *Geometric Quantization*, 2nd Ed. (Clarendon Press, Oxford, 1992).
- [51] T.T. Wu and C.N. Yang, Concept of nonintegrable phase factors and global formulation of gauge fields, *Phys. Rev. D* 12 (1975) 3845–3857.